







**Proceedings of the 20th Annual National Congress of the  
Association for Mathematics of South Africa**

**Volume 1**

*Demystifying Mathematics*

07 - 11 July 2014

Diamantveld High School  
Kimberley

**Editors:** Mandisa Lebitso and Anne Maclean





Copyright © reserved

Association for Mathematics Education of South Africa (AMESA)

P.O. Box 54, Wits, 2050, Johannesburg

*Proceedings of the 20<sup>th</sup> Annual National Congress of the Association for Mathematics of South Africa, Volume 1, 07 to 11 July 2014, Kimberley.*

All rights reserved. No reproduction, copy or transmission of publication may be made without written permission. No paragraph of this publication may be reproduced, copied or transmitted, save with written permission or in accordance with the Copyright Act (as amended). Any person who does any unauthorized act in relation to this publication may be liable for criminal prosecution and civil claim for damages.

**First published: July 2014**

**Published by AMESA**

**ISBN: 978-0-620-61475-7**

## FOREWORD

The theme of the 2014 AMESA Congress is: ‘Demystifying mathematics’.

Why the need to demystify mathematics?

When was the last time you got up from your sofa to change the channel on your TV set or made a call from a public pay-phone or posted a letter to a friend? The world is changing at a rate as never before.

In the 21st century, scientific and technological innovations have become increasingly important as we face the benefits and challenges of both globalization and a knowledge-based economy. To succeed in this new information-based and highly technological society, students need to develop their capabilities in STEM to levels much beyond what was considered acceptable in the past.

Added to this, is the growing global recognition of the urgency of tackling a range of difficult and complex issues that impact on our human well-being. The world’s population is estimated to rise from 7 billion to 8 billion by 2030 and this, against the backdrop of declining resources and changing climate conditions, means re-shaping where we live and how we live. Finding a way to adapt, to be efficient and sustainable, will require knowledge from many different sources.

In the South African context, where the number of learners taking mathematics and physical sciences at school level is declining, and the quality of matric results in these two gateway subjects is catastrophic, the pool from which the country is to grow its knowledge-based economy is less than sufficient.

South Africa has the third highest unemployment rate in the world for people between the ages of 15 to 24, according to the World Economic Forum (WEF) Global Risk 2014 report. The report estimates that more than 50% of young South Africans between 15 and 24 are unemployed. The quality of schooling, in particular, numeracy and mathematics competency, is closely linked to unemployment.

If we are to ensure a healthier economy for South Africa, better living conditions for all South Africans and a greener and more prosperous world for future generations, then the solutions have to be found in how we respond to ‘demystifying’ mathematics.

We hope that the presentations and deliberations at the 2014 congress, as well as the papers in the Proceedings, will go a long way towards addressing these pressing issues.

Mandisa Lebitso and Anne Maclean

July 2014



## REVIEW PROCESS

The papers accepted for publication in this volume of the Proceedings (*Long Papers and short papers*) were subjected to triple-blind peer review by three experienced mathematics educators. The academic committee considered the reviews and made a final decision on the acceptance or rejection of each submission, as well as changing the status of submission. Authors of accepted submission were given the option of submitting an extended abstract rather than their full submission for publication in the publication elsewhere..

Number of submissions:	116
Number of plenary paper submissions:	5
Number of long paper submissions:	30
Number of short paper submissions:	11
Number of workshop submissions:	43
Number of 'How I teach' paper submissions:	14
Number of poster submissions:	0
Number of submissions accepted:	103
Number of submissions rejected:	4
Number of submissions withdrawn by authors:	9

We thank the reviewers for giving their time and expertise to reviewing the submissions.

### Reviewers:

Jogy Alex	Zingiswa Jojo	Mdutshekelwa Ndlovu
Sarah Bansilal	Karen Junqueira	Marc North
Anita Campbell	Erna Lampen	Craig Pournara
Pam Fleming	Pamela Lloyd	Ingrid Sapire
Faaiz Gierdien	Caroline Long	Jackie Scheiber
Nico Govender	Kakoma Luneta	Sibawu Witness Siyepu
Rajendran Govender	Judah Makonye	Avhasei Tsanwani
Diliza Hewana	Duncan Mhakure	Anelize van Biljon
Mark Jacobs	Alfred Msomi	Lyn Webb
Shaheeda Jaffer	Jayaluxmi Naidoo	

## TABLE OF CONTENTS

---

### Plenary Papers

---

Rajendran Govender	Generalizations in mathematics: from primary to secondary school and beyond	1
Andy Green	The power of mathematics at 1000 mph	4
Anne Maclean	Demystifying mathematics: <i>Let's keep our world alive</i>	11
Vimolan Mudaly	So, what do our teachers know?	29
Ingrid Rewitzky	Mathematical structures, and tight connections between them	31



---

## Long Papers

---

Stanley Adendorff	Conversions in the metric system: Reflecting on the get mathematics curriculum	33
Robyn Clark	Exploring learning trajectories in algebra: informing the teaching of functions	44
Joseph J Dhlamini	A review of research on the effect of using a group learning approach in mathematics classrooms	46
Roland Fray	Exploring teachers' notion of logical thinking	58
VG Govender	Factors contributing to the popularity of Mathematics Olympiads and Competitions in some schools: An interrogation of learners' and teachers' views	70
VG Govender	Using the South African Mathematics Challenge to develop pre-service Mathematics teachers' problem-solving abilities	87
Zingiswa M. Jojo, Joseph J. Dhlamini, M.M. Phoshoko and M.G. Ngoepe	Exploring the mathematical proficiency of Grade 6 teachers: A case of Gauteng Tswane East	105
Sbongile Mashazi	Learners' explanations of the errors they make in introductory algebra	118
Bruce May	Deepening thinking-like problems: The case of two students	130
Monde Mbekwa and Rajendran Govender	A mathematical problem-solving exercise by in service teachers in a continuous teaching development context	140
Duncan Mhakure, Marie Jacobs and Cyril Julie	Grade 10 students' facility with rational algebraic fractions in high-stakes examination: observations and interpretations	151
Vimolan Mudaly	Diagrams: Are they useful in mathematics?	164
Benadette Aineamani and Seshni Naicker	Mathematics textbook analysis: A guide to choosing the appropriate mathematics textbook	165
Jayaluxmi Naidoo	Exploring the use of activity theory as a framework for teaching & learning of mathematics	176
Bharati Parshotam and Erna Lampen	Construction routes in the solution of compass and straightedge construction problems: Geometric reasoning of first year mathematics education students	189

Kim Ramatlapana	Developing rubrics for TPACK tasks for prospective mathematics teachers: A methodological approach	198
Dave Rowley	Data is key	211
Sibawu Witness Siyepu and Thembinkosi Mtonjeni	Geometrical concepts in real-life context: A case study in South African Traffic Road Signs	213
Charles Smith and Cyril Julie	Question types in mathematics	223
Erica D. Spangenberg	A comparison of the achievement goal orientation of mathematics learners WITH/WITHOUT attention deficit hyperactivity disorder (ADHD)	234
Marié Weitz	Pedagogic actions and strategies that support sophistication within Foundation Phase number work	246



---

## Short Papers

---

Benadette Aineamani	The role of teachers in developing learners' mathematics' mathematics discourse	256
Ryna du Plooy and Caroline Long	Engaging with cognitive levels: A practical approach towards assessing the cognitive spectrum in mathematics	267
Caroline Long and Erna Lampen	Teacher agency and professional practice: Developing and nurturing creativity in education	275
Sello Makgakga	Error discourse analysis in solving quadratic equations by completing the square	284
J. P. Makonye, Marie Weitz and Bharati Parshotam	Preparing to teach Grade 12 financial mathematics: A case study of student teachers' learning experiences	293
Shauna Pamplin and Nicky Roberts	Managing and leading mathematics improvement in Foundation Phase	299
Vasen Pillay, Mampotse Shirley Ramasia and Taona Nyungu	Teaching with a 'critical feature' of an 'object of learning' in focus	301
Craig Pournara	Effective interest rates: Making sense or cents?	310
Helene Schoeman; Tania Halls and Nicky Roberts	How we teach time in Grade 3	317
Nkosingiphile Shelembe and Sibongile Mashazi	Lesson Plan Study on how to choose examples and prepare explanations for hyperbola graph in Grade 10	320
Avhasei Richard Tsanwani	Connecting mathematics with its history	330

# GENERALIZATIONS IN MATHEMATICS: FROM PRIMARY TO SECONDARY SCHOOL AND BEYOND

Rajendran Govender

University of Western Cape

## *Extended Abstract*

The development of generalizations plays a pivotal role in fostering both mathematical thinking and mathematical growth. According to Mason (1996) generalization is the heart and soul of mathematics. If teachers are unaware of its prevalence and promise, and not in the habit of getting their learners to experiment, make conjectures, express and justify their own generalizations, then mathematical thinking becomes the worst casualty of our mathematics classroom. Furthermore, any form of justification supporting a generalization (like: external conviction, empirical, generic and deductive) invokes a particular kind of reasoning or a combination of reasoning forms (like inductive, analogical, deductive) which inevitably provides the grounds on which students can naturally question, argue and conjecture, construct a generalization, and/or also explain why a particular generalization is either true or false (compare Blanton & Kaput, 2002; De Villiers, 2003a; Ellis, 2007; Hanna, 2000).

With specific reference to learning and doing mathematics, the National Statement on Mathematics for Australian Schools contends that

“Mathematical discoveries, conjectures, generalizations, counter-examples, refutations, proofs are all part of what it means to do mathematics. School mathematics should show the intuitive and creative nature of the process, and also the false starts and blind alleys, the erroneous conceptions and errors of reasoning which tend to be a part of mathematics” (Australian Educational Council, 1991, p.14).

Indeed the cyclical processes of experimentation, conjecturing, testing, generalizing, refuting and justifying that mathematicians traverse in order to construct the polished definitions and theorems that we find in most mathematics textbooks and curriculum documents around the world are not reported on in most cases (compare De Villiers, 2004, 2010). In particular, Freudenthal (1973) talks about the way in which textbooks hide, disguise or distort the way in which real mathematics is invented, and thus argues for an approach of ‘re-invention’. Taking cognisance of all this, the new South African mathematics Curriculum through all the grades places appropriate emphasises on reasoning, conjecturing, generalizing and justifying across the content topics.



For example, the Curriculum Assessment Policy Statement (CAPS) Mathematics Senior Phase Grades 7-9, teachers are expected to give learners chances at investigations that are designed within the context of the following broad guidelines:

“Investigations promote critical and creative thinking. It can be used to discover rules or concepts and may involve inductive reasoning, identifying or testing patterns or relationships, drawing conclusions, and establishing general trends.” (p. 156).

This presentation provides plausible ideas as to how learners can construct inductive generalizations and experience heuristic counter-examples (wherever possible) that can cause (or force) them to modify (refine) their conjectures (or inductive generalizations), i.e. make a conceptual change as per Piaget’s model of socio-cognitive conflict. In addition, ways of justifying an inductive generalization are explored and discussed.

Furthermore, this presentation provides a road map of how analogical reasoning can be used to extend generalizations from one domain to the next, as well as aiding in the construction of deductive generalizations and logical explanations of particular generalizations through parallel transfer of explanatory structures from one domain across to another domain. In particular, this presentation brings to the fore three basic ways in which generalizations can be developed: namely inductive, analogical and deductive.

Establishing a learning environment in our mathematics classrooms, wherein students are given an opportunity to experience the underlying processes of experimenting, conjecturing, specializing, generalizing and justifying can enable mathematics learners to come to view mathematics as a process that is within their capabilities and not just a series of ‘products’ that are produced, and hence learn mathematics in a more meaningful way.

## REFERENCES

- Australian Educational Council. (1991). *A National Statement on Mathematics for Australian Schools*. Carlton, Victoria, Austria: Curriculum Corporation.
- Blanton, M., & Kaput, J. (2002, April). *Developing elementary teachers’ algebra “eyes and ears”*: Understanding characteristics of professional development that promote generative and self-sustaining change in teacher practice. Paper presented at the annual meeting of the American Educational Research Association, New Orleans, LA.
- De Villiers, M. (2003). *Rethinking proof with Geometer’s Sketchpad 4*. Emeryville, CA: Key Curriculum Press.
- De Villiers (2004). The Role and Function of Quasi-empirical Methods in Mathematics. *Canadian Journal of Science, Mathematics and Technology Education*, 4(3), 397-418. doi: [10.1080/14926150409556621](https://doi.org/10.1080/14926150409556621).

- De Villiers (2010). Experimentation and proof in Mathematics. In G. Hanna, H.N.Janke & H.Pulte (Eds.), *Explanation and Proof in Mathematics: Philosophical and Education Perspectives* (pp. 205-221). New York: Springer.
- Ellis, A.B. (2007). Connections between generalizing and justifying: Students reasoning with linear relationships. *Journal for Research in Mathematics Education*, 38(3), 194-229.
- Freudenthal, H. (1973). *Mathematics as an educational task*. Dordrecht: D. Reidel.
- Hanna, G. (2000). Proof, explanation and exploration: An overview. *Educational Studies in Mathematics*, 44(1), 5-23.



## THE POWER OF MATHEMATICS AT 1000 MPH

Wing Commander Andy Green OBE (Mathematician)

Project BLOODOUND

The theme of the AMESA Congress 2014 is demystifying mathematics. The challenge we face when working with learners of all ages, in particular grades 4 to 9, is how to make the subject interesting, exciting and even **fun** in the classroom. Sadly, the result of this failure to excite is a worldwide shortage of scientists, engineers and mathematicians who will need to come up with answers to the global challenges we face that include shortages of energy, clean water and food, health, security and global warming to name just a few.

There was a terrific increase in the number of science and maths graduates worldwide as a result of the US moon landings programme of the late 1960's and early 1970's where youngsters were fascinated by the enormous challenges answered by the scientists and engineers at NASA. The BLOODHOUND education programme would like to create a mini Apollo effect and create a surge in learners wanting to become the next generation of scientists and engineers. This is the number one aim of the BLOODHOUND Project, and not surprisingly, to break the current World Land Speed Record (WLSR)! In fact it was a requirement set by the UK government that we develop an education programme to overcome the countries shortage of scientists and engineers. They would then loan us the world's most powerful lightweight jet engine, the Eurojet EJ200, and in itself an amazing piece of engineering jewellery.

The BLOODHOUND SSC project was launched in October 2008 with the engineering objective of designing, building and running a car to achieve a new WLSR of 1000 mph. This engineering objective is coupled with the primary objective to promote science, technology, engineering and mathematics (STEM) to schoolchildren in the UK and beyond via the BLOODHOUND education programme. Clearly, the aerodynamic challenges associated with developing a land-based vehicle capable of safely achieving speeds of up to 1 690 km/h (1 050 mph and approximately Mach 1.3) are great, particularly at supersonic speeds and only ever achieved previously by one vehicle – ThrustSSC in 1997 – 17 years ago! Drag minimisation and vertical aerodynamic force control are of paramount importance for a safe record attempt on the constrained distance of 20 km available at the record attempt site of Hakskeenpan in the Northern Cape. Computational fluid dynamics (CFD) were chosen as the primary tool to guide the aerodynamic design of the vehicle as was the case in in the 1990's when the mathematical technology was in its infancy.

The CFD system used for the BLOODHOUND study was the Swansea University FLITE3D system. This is a classical cell-vertex finite volume method, with stabilisation and discontinuity capturing, implemented on hybrid unstructured meshes. The next few paragraphs briefly describes the CFD process used to analyse such a complex aerodynamic problem but its focus is on the predicted aerodynamic behaviour of the final frozen design (known as configuration 12 – abbreviated to config12) of BLOODHOUND SSC. It outlines both the on and off-design aerodynamic characteristics of the vehicle and considers the impact of different engine conditions and directional stabilities.



Figure 1: 2007 artist's impression of the 'yet to be named' BLOODHOUND World Land Speed Record vehicle.

### **The BLOODHOUND SSC aerodynamic problem**

In the past three decades, CFD has revolutionised the way in which the aerospace industry tackles problems of aerodynamic design. In particular, unstructured mesh methods now allow grids on complex 3D geometries to be generated in a matter of hours, which might once have taken several months using multi-block techniques. In light of this, CFD has become an integral part of the typical aerodynamic design cycle.

On the BLOODHOUND SSC project, for practicality, financial restrictions and time constraints, the major design cycle loop has focused on the actual vehicle runway and desert testing. This is where validation of the CFD modelling used will take place. It will be the case that initial vehicle runs on Hakskeenpan in 2015 will lead to significant re-design work ready for the target speed of 1,000 mph in 2016.



The major design changes that have taken place during the design cycles as a result of aerodynamic modelling are shown in Figure 2 below.

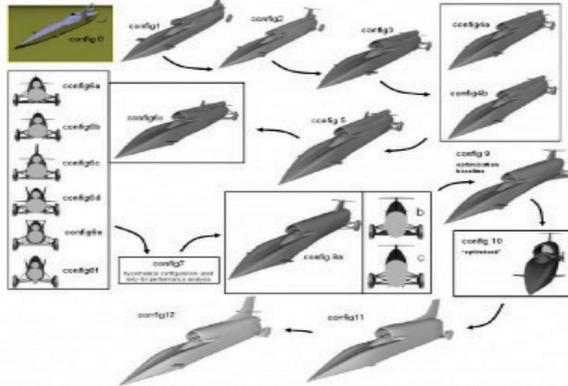


Figure 2

1. Between config 0 and config 1, a transition was made from a design with a twin (bifurcated) jet intake to a design with a single jet intake. This change was made to improve the ‘quality’ of the flow at the jet compressor face across the Mach range. The front wheels also moved from a staggered configuration to a symmetric configuration.<sup>15</sup>
2. Between config 1 and config 4, the positioning of the rear winglets and the vertical fin was varied, and the rear suspension was designed and included in the CFD model.
3. Between config 4 and config 6, the rear-wheel suspension geometry was varied together with the body shaping as a result of internal packaging constraints. The size and area distribution of the jet intake were also varied.
4. Config 7 existed only as a hypothetically perfect vehicle from an aerodynamic performance point of view. These hypothetical data were used in the overall vehicle performance model as a baseline case to give an indication of criteria such as the required track length for vehicle testing.
5. Config 8 introduced the rear ‘delta fairing’<sup>22</sup> for supersonic lift minimisation at the rear of the vehicle.
6. Between config 9 and config 12, a series of parametric optimisation studies was carried out together with an increase in the fin size to achieve the directional static margin target.

The BLOODHOUND design engineering team was setting out to do something truly extraordinary. This wasn't just designing a new airliner, for example, where they could build on vast amounts of prior knowledge (Comet, Boeing 747, 777 or Dreamliner, Airbus A350 or A380), by making marginal gains to create the next generation. The sheer audacious ambition of increasing the current WLSR by over 30% meant that they had to start from a blank sheet of paper and not only design a new type of land speed record vehicle, but also develop a whole new way of design thinking.

Of course, the question of '*can we keep the car on the ground?*' was an early topic of conversation, but they could never have guessed that this problem would cause far more of a headache at the rear of the car, than just keeping the nose down at the front. This unforeseen aerodynamic behaviour led to the six month rear suspension optimisation study that resulted in the 'delta fairing' design which has been a feature since 'config 9'. In those early front room conversations they hadn't anticipated that getting the twin intake bifurcated duct in the original design to deliver a suitable flow to the EJ200 compressor face across the entire Mach range would be so difficult - another mathematical design challenge for the team. Of course, this eventually led to revert to a single intake above the cockpit canopy, causing significant structural headaches to be overcome. In those early days the team had no real 'feel' for how directionally stable the car would be, which in turn meant they didn't really have an idea of how big the fin would need to be to 'keep the pointy end heading forwards'.

In fact, for the first few iterations of aerodynamic design, where they almost completely focused on the question of what the external shape of the vehicle should be like, but still answering the question '*is 1000 mph even possible?*', They were constantly being surprised by the aerodynamic performance that the CFD simulations were predicting. That was a little nerve-wracking for the team and the driver! Of course there were things the team could do to investigate the causes of odd aerodynamic phenomena, such as flow visualisation. This led to some of the famous BLOODHOUND imagery that has been available over the years (e.g. Figure 3) and force distribution breakdowns. This has allowed the team (in the virtual world) to gain a far better understanding of how BLOODHOUND SSC will behave when testing commences next year.



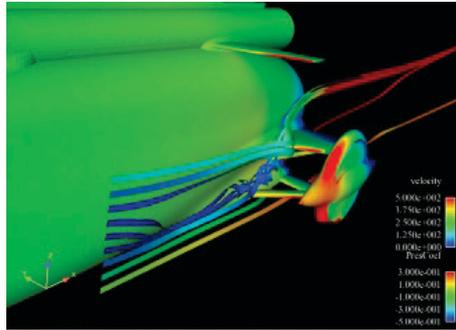


Figure 3: CFD flow visualisation of the complex flow around the rear wheels of BLOODHOUND 'config 4'

The aerodynamic team has really been on an astounding journey of engineering design research. Figure 2 shows the journey of design evolution from 2007's 'config 0' through to the 'config 12' design that is currently being built in Bristol in south west England. One thing that you should be able to spot from this view of the design evolution is that they were 'homing in' on the optimum shape where the geometric shape changes get smaller and smaller. Anyone who has used any form of trial and error, which is essentially what we do in engineering design, will be familiar with this pattern. But, more importantly, what else has been happening is that the aerodynamic effect of making changes to the geometric exterior has become more and more predictable.

In fact, with the most recent and subtle changes to the exterior of the vehicle, Ron Ayers and Ben Evans have been able to confidently predict the impact on aerodynamic performance intuitively and have then used CFD simulations as a 'double check'. For a mathematician this is a much happier position to be in than the early days of 'running the CFD blind' and then crossing fingers that we could make sense of the data that came back. CFD and intuition are now working together in partnership!

Of course, the question, now that we leave the virtual world behind and turn our eyes towards running in the reality of testing on Hakskeenpan, is whether or not this *path to predictability* will continue.

This insight into the aerodynamic design challenges faced by the engineering team has been one major chunk of the jigsaw, but one that had to be completed prior to any build could take place. Hence the delay of four years in coming to South Africa. As you can imagine the driver was reluctant to sit in the car and attempt the WLSR without confirmation it would stay on the ground and travel pointy end first!

There has also been major research carried out on a car's component that most of us take for granted - the wheels! The four wheels on BLOODHOUND SSC are 90 cm diameter and have a mass of 95 kg each. They rotate at 10 300 rpm, that's 171 revolutions per second and generate 50 000 radial g at the rim. If you were to place a 1 kg bag of sugar on the rim, at 1,000 mph that bag of sugar would weigh 50 000 kg. So just with the wheels there is very exciting mathematics involved that has kept major universities and international companies occupied for some time! Plus the engineers needed to consider what would happen if a small stone from the pan surface kicked up and collided with the car at supersonic speed. So they had to investigate the ballistics of such incidents and the gas gun at Cambridge University came in very useful.

So what else was going on with the mathematics and science associated with travelling at very high speeds? Well think about what happens to air flowing around the sides of the car at high speed. Above approximately Mach 0.3 (a third of the speed of sound, or roughly 320 km/h) air begins to take on the important property of compressibility. This means that air can not only change direction to navigate around an obstacle, but it can now also 'squeeze' or compress to help it on its way. The extent to which this compression, and associated density rise occurs, depends upon the speed of the air relative to the speed of sound and the shape of the object it is trying to make its way around. The reason BLOODHOUND SSC is long and thin is to try and reduce the extent of this compression as this usually translates into drag. The ultimate expression of air compression is a shock wave, within which this compression occurs almost instantaneously and manifests itself in the supersonic 'BOOM' and the beautiful pictures of the THRUSTSSC, see Figure 4



Figure 4 – The supersonic shock wave emanating from the nose of ThrustSSC and showing up late in the afternoon intensified by a dust storm earlier in the day.



### The mathematics of science

Now imagine yourself as an innocent particle of air in the path of BLOODHOUND SSC travelling towards you at 1 610 km/h (roughly 1.3 times the speed of sound). First of all you will have to navigate around the nose of the car by squeezing/compressing/slowing down through a shock wave (the bow shock on the car). Once you have successfully avoided the nose, you can then, briefly, accelerate down the side of the car until you meet air coming at you from above the cockpit canopy that itself has had to ‘shock’ and squeeze past the canopy and upcoming jet intake duct. This has caused you to have to change path, compress again in a shock wave and slow down. You have now survived this second interruption to your day and you are given the opportunity to accelerate down the side of the rear half of the vehicle until something rather ominous approaches you – one of the rear wheels. Just when you thought you were home and dry, you have to experience a third shock, which turns out to be the strongest shock of all. The density of the air through this shock wave almost doubles, and it is this that gives rise to high pressure at the front of the rear wheel fairings and also the source of our huge rear supersonic lift.



Figure 5. BLOODHOUND SSC config10 top – note the long and slender shape to minimise air compression and hence drag.

All we can provide at the AMESA Congress is a brief insight into the leading edge research associated with a car designed to travel at 1 690 km/h. Almost every education subject can be enhanced by using BLOODHOUND as an exciting context for learning. The choice of Hakskeenpan, the culture and history of the local Mier people, how to develop tourism opportunities, writing press releases and using dramatic language to describe the drivers experience – all opportunities for educators to use BLOODHOUND in the classroom. The challenge that educators now face is how to expose as many learners as possible to the excitement of engineering, which is based so centrally on mathematics. Teachers are the key to economic success and prosperity in every country.

## DEMYSTIFYING MATHEMATICS: *LET'S KEEP OUR WORLD ALIVE*

Anne Maclean

Managing Director, MATHS & SCIENCE LEADERSHIP ACADEMY NPC,  
KIMBERLEY

### INTRODUCTION

The theme of the 2014 AMESA Conference is: 'Demystifying Mathematics'. According to the MacMillan Dictionary, the thesaurus entry for "**demystify**" is as follows: To make something that is difficult easy to understand, especially by explaining it clearly.

If one looks at mathematics education in South Africa, there seem to be serious challenges in making mathematics easy to understand with catastrophic implications for the growth of South Africa's economy and poverty reduction.

In this paper I intend to highlight a few of the challenges facing mathematics education in South Africa as reported on by researchers and education specialists.

Can we go on like this?

What is 21<sup>st</sup> century teaching and learning calling us to?

The Maths & Science Leadership Academy (MSLA), in partnership with the Northern Cape Department of Education and various other partners, is playing a vital role in Kimberley in addressing some of the challenges facing maths and science education and in equipping youth for success in the 21<sup>st</sup> century.

It is hoped that through sharing good practices and MSLA's vision for the future, we may grow the circle of excellence.

As inspired mathematics teachers:

- *let's rekindle the fire of passion for what we do*
- *let's strive to become world class maths teachers*
- *let's take ownership where we have failed*
- *let's change what can be changed*
- *let's ensure a bright future for our youth*
- *let's acknowledge the importance of our role as mathematics teachers in growing the economy and reducing poverty levels*

*Together, let's start building the world we want - let's keep our world alive!*



## SA's MATHS CHALLENGES

According to a press release by the South African Institute of Race Relations:  
**NOT ADDING UP: TOO FEW MATHS TEACHERS TO SATISFY DEMAND,**  
J Snyman:

A total of 84 high schools across the country did not offer mathematics for grade 10, 11 and 12 in the 2012 academic year primarily as a result of a shortage of suitably qualified maths teachers.

... The number of pupils taking mathematical literacy now outstrips the number of pupils taking mathematics. In 2008 (the year the NSC was introduced), 35 000 more pupils took mathematics than took mathematical literacy. By 2012 this ratio had reversed with 65 000 more pupils taking maths literacy than mathematics.

Dr Martin Prew, a visiting fellow at Wits University's school of education and an education development specialist is of the view that:

Schools keep pass rates up by limiting subject choices - sacrificing our poorest pupils' futures.

... It appears that, in pursuing the goal of high pass rates, pupils' life chances are being stunted. This, at best, is an unintended consequence of central pressure to be able to boast being the best-performing province in the country. As a result universities cannot train adequate engineers, South African industry and commerce are starved of skills they desperately need and the development of the country is subsequently slowed.

The Centre for Development and Enterprise (**CDE**) is one of South Africa's leading development think tanks, focusing on critical development issues and their relationship to economic growth and democratic consolidation.

CDE INSIGHT (October 2013), focuses on: **MATHEMATICS OUTCOMES IN SOUTH AFRICAN SCHOOLS**. What are the facts? What should be done?

This report by J McCarthy and R Oliphant is a summary of two specially commissioned research papers for CDE, both independent studies of the state of schooling in South Africa as of early 2013, carried out by university-based experts N Spaull and C Simkins.

A brief summary of the report is as follows:

South Africa is significantly underperforming in education, particularly mathematics teaching and learning. Mathematics teaching is often poor quality, with teachers not able to answer questions in the curriculum they are teaching, one indicator of the challenge. Often national testing is misleading as it does not show the major gap at lower grade levels. Of the full complement of pupils who start school, only 50 percent will make it to Grade 12 and only 12 percent will qualify for university entrance. Fundamental reforms are needed in the public sector.

Business leaders need to incorporate an understanding of private education and other market experiments and schooling innovations in their overall perspective and priorities for intervention and reform. (Page 1)

The report makes specific reference to the findings in the 2011 Trends in International Mathematics and Science Study (TIMSS), highlighting the teaching of mathematics in South African schools as being amongst the worst in the world and citing teacher complacency as being a major problem in addressing the situation.

The 2011 TIMSS showed that South Africa performed worse than any other middle-income country. The average South African Grade 9 learner is 2 years' learning behind the average Grade 8 learner from 21 other middle income countries in mathematics (and 2,8 years behind in science).

... In the recent TIMSS 2011, 89 percent of South African Grade 9 teachers felt 'very confident' in teaching mathematics, in stark contrast to teachers in Finland (69 percent very confident), Singapore (59 percent very confident) and Japan (36 percent very confident), the best performing countries. This is particularly at odds with Grade 9 student performance, where 32 percent of South African students perform worse than random guessing on the multiple choice questions. (Page 7)

**THE CLASS OF 2013 ACHIEVED THE HIGHEST MATRIC PASS RATE (78.2%) SINCE 1994.**

In the article, **BEHIND THE MATRIC RESULTS: THE STORY OF MATHS AND SCIENCE**, by G Campbell and M Prew, great concern is expressed about the performance in these two gateway subjects. The two critical issues being: the drop in the number of learners writing these subjects and the quality of passes achieved.

Several international studies and the Annual National Assessment (ANA) results indicate that the problem with mathematics has its roots in primary school, where many learners fail to gain basic mathematical skills.

The 2013 ANA results showed only 39% of Grade 6 learners and 2% of Grade 9 learners achieving above 50% in mathematics.

Learners are not taught the basics in Grade 7 - 9 (Senior Phase) as it is assumed learners have mastered these basic mathematics concepts by the end of Grade 6.

**ACCORDING TO THE REPORT, THE REASON FOR THE DROP IN PERFORMANCE IN GRADE 9 IS:**

a direct consequence of compounding backlogs and increasingly inventive ways learners use to beat the system and dodge detection, with dire consequences for individual learners and the system.



## HOW IS THE PROBLEM DEALT WITH AT SECONDARY SCHOOL LEVEL?

Secondary schools often deal with the problem in two ways. They try and stop all but the most able learners selecting to do mathematics and physical science in grade 10, and also hold back (or "warehouse", as some teachers refer to the practice) learners in the mathematics stream who fail grade 10 or grade 11 exams.

Then, in grade 11 and 12, teachers attempt to ensure that the NSC results are good by cramming and question-spotting in extra classes, Saturday schools and through private tutor programmes. This often leaves learners with the ability to pass the NSC but with no depth to their knowledge – a fact that is only discovered at tertiary level.

Figures from the Department of Basic Education show that of the 562 112 full-time candidates who wrote the National Senior Certificate (NSC) in 2013, only 43% sat the maths paper.

A total of 97 790 of these students (40.5%) achieved a pass mark in mathematics of 40% and above.

Just 26.1% achieved a pass mark of 50% and above.

A dismal 15.6% achieved 60% and above.

According to the 2013 Organisation for Economic Co-operation and Development (OECD) Country report on South Africa:

... To redress ‘catastrophically high unemployment among the youth... [51 percent in the fourth quarter of 2012] ...education remains... the critical problem’.

The OECD argues that South Africa’s educational outcomes are ‘aggravating the excess supply of unskilled labour and worsening income inequality. (CDE Insight, October 2013. Page 10)

## CAN WE GO ON LIKE THIS?

### TEACHING AND LEARNING FOR THE 21ST CENTURY

**Andreas Schleicher** (OECD Education Directorate), in his article: THE CASE FOR 21ST- CENTURY LEARNING, inspires a new way of thinking – a call to change of mind set – if we are to prepare our learners to be successful in the 21<sup>st</sup> century. May we take his powerful message to heart and, indeed, into our maths classrooms.

Our world is fast changing and so, too, must our way of teaching and learning change if we are to address the challenges of the future.

Schleicher states that there are two facts why knowledge and skills are important to the future of our economies.

Firstly, from a jobs perspective, it pays to study.

21<sup>st</sup> century learning, however, goes deeper than skilling up.

It is about how knowledge is generated and applied, about shifts in ways of doing business, of managing the workplace or linking producers and consumers, and becoming quite a different student from the kind that dominated the 20th century. What we learn, the way we learn it and how we are taught is changing. This has implications for schools and higher level education, as well as for life-long learning.

Is the call, then, to become more business-like in our approach to how schools should function?

If you were running a supermarket instead of a school and saw that 30 out of 100 customers each day left your shop without buying anything, you would think about changing your inventory. But that does not happen easily in schools because of deeply rooted, even if scientifically unsupported, beliefs that learning can only occur in a particular way.

We live in a fast-changing world, and producing more of the same knowledge and skills will not suffice to address the challenges of the future. A generation ago, teachers could expect that what they taught would last their students a lifetime. Today, because of rapid economic and social change, schools have to prepare students for jobs that have not yet been created, technologies that have not yet been invented and problems that we don't yet know will arise.

Schleicher asks the question: How do we foster motivated, dedicated learners and prepare them to overcome the unforeseen challenges of tomorrow?

It goes about what education should be in today's world.

... educational success is no longer about reproducing content knowledge, but about extrapolating from what we know and applying that knowledge to novel situations.



Education today is much more about ways of thinking which involve creative and critical approaches to problem-solving and decision-making. It is also about ways of working, including communication and collaboration, as well as the tools they require, such as the capacity to recognise and exploit the potential of new technologies, or indeed, to avert their risks. And last but not least, education is about the capacity to live in a multi-faceted world as an active and engaged citizen. These citizens influence what they want to learn and how they want to learn it, and it is this that shapes the role of educators.

The development of imaginative skills is important for the 21<sup>st</sup> century and requires that we move away from teaching students the techniques of solving manageable bits and pieces of problems. Schleicher makes the point that today knowledge demands open-mindedness, making connections between ideas and becoming familiar with knowledge in other fields. It is through learning and teaching across disciplines that future inventions, and probable sources of economic value, will come from.

Numeracy and mathematics are not the only challenges that South Africa's education is facing. According to Nick Taylor, the head of the unit which reports directly to Basic Education Minister Angie Motshekga, the literacy level of South African Grade 5 pupils is a "national catastrophe".

According to Schleicher: Rather than just learning to read, 21st century literacy is about reading to learn and developing the capacity and motivation to identify, understand, interpret, create and communicate knowledge.

It is very important that students are encouraged to learn on their own and to enrich their knowledge through communication and collaboration.

Innovation in particular is the outcome of how we mobilise, share and link knowledge.

Schleicher concludes:

Success will go to those individuals and countries that are swift to adapt, slow to resist and open to change. The task for educators and policymakers is to help countries rise to this challenge.

### **THE MATHS & SCIENCE LEADERSHIP ACADEMY NPC**

The Maths & Science Leadership Academy (MSLA) was initiated by Anne Maclean, whilst in the employ of the Northern Cape Department of Education, in 2006, in direct response to the challenges facing maths and science education in the Kimberley area. The non-profit organisation started from very humble beginnings but through the establishment of partnerships with various stakeholders in education, (private sector, government and civil society), it is fast becoming a model of success.

Kimberley is the capital of the Northern Cape Province, the largest province in South Africa, with the smallest population and which faces the greatest challenges: far distances, severe climate (very hot in summer and very cold in winter), arid conditions and very few resources.

*A special goal of the Maths & Science Leadership Academy is to bring hope to the youth of the Northern Cape and to motivate them to have a dream and to DREAM BIG!*

*The majority of youth in this province are faced with serious challenges:*

- *Uninspiring and unsafe school environments in the poorer areas*
- *High rate of unemployment*
- *Poor socio-economic circumstances*
- *No recreation facilities*
- *High rate of teenage pregnancy*
- *High rate of alcohol and substance abuse*
- *Very few students at tertiary institutions*
- *No role models*

*In response to the challenges mentioned above, and to those mentioned earlier, the MSLA maths & science programmes currently aim to:*

- motivate, support and empower teachers and student teachers
- develop a passion for maths and science in learners from an early age
- increase the number of learners who opt for maths and science in the FET Phase
- increase the number of learners who achieve more than 50% for mathematics and science at matric level
- produce students who will be a success at tertiary level in fields that are critical to growing the economy
- provide opportunities for learners to develop the skills and knowledge needed for success in the 21<sup>st</sup> century
- grow and develop future leaders of calibre for a brighter and greener South Africa

It is not enough to merely raise the level of academic achievement if we are to grow tomorrow's leaders for a brighter and greener South Africa. We therefore focus on the holistic development of our youth and provide them with novel opportunities to develop skills needed in the 21<sup>st</sup> century, namely: communication, collaboration, problem-solving, innovation, research and IT.

When it comes to leadership, it is all about growing the MADIBA legacy – learners seeing themselves as young leaders who are knowledgeable about national and global challenges and who have the values, desire and passion to work towards the future this world wants.



## MSLA MATHS & SCIENCE PROGRAMMES

### GRADE 9 – 12 MATHS & SCIENCE PROGRAMME

While maths & science interventions across the country tend to focus more on Grade 12 learners for the sake of improving matric results, the MSLA approach is to target learners in the lower grades and to provide them with as many opportunities as possible to develop holistically until they have completed their schooling..

MSLA currently provides 270 Grade 9 – 12 learners from 20 schools in Kimberley (mainly from the poorer areas) with extra academic support in maths and science through afternoon classes, holiday academies and the Saturday Maths Pi-oneers Programme. The focus of the holiday academies also includes English / Afrikaans, as well as novel challenges aimed at developing leadership skills and exposing the learners to global issues.

Each year the organisation secures a sponsor to “adopt” a new group of Grade 9 learners with the expectation that the same learners will be supported until they have completed matric. The learners receive much; therefore much is expected in return. Hard work, commitment, good results and involvement in community service are the basic requirements if learners are to remain in the programmes.

The afternoon maths and science classes are run at the MSLA Campus, a De Beers owned heritage building established in 1901, which consists of a few office areas, a classroom, a science lab, a small computer lab, ablutions and a tiny kitchen area.

The majority of learners are provided with transport from their schools to the campus and home again after their classes which run from 15:00 – 18:00, Monday to Thursday during school terms. Each learner attends one afternoon session per week. They are taught by the best maths and science teachers available and are provided with stationery and study materials, as well as refreshments. At the end of term 1 and term 3 they all write a test on the work covered in maths and science which is aligned to CAPS. These results, together with their school June and November results are used to gauge their progress and to report to sponsors.

### MSLA MATHS PI-ONEERS PROGRAMME

The MSLA Maths Pi-oneers Programme focuses on developing maths problem-solving skills in learners and on developing teachers’ *competence and confidence in teaching this important subject, through the use of Maths Olympiad questions.*

The programme currently focuses on the GET and FET phases. There are 225 learners and 18 teachers in the Grade 4 – 6 programme and 400 learners, 3 student teachers and 20 teachers in the Grade 7 – 12 programme. Each programme runs over 5 Fridays and Saturdays throughout the year.

The teacher training sessions take place at the MSLA Campus from 14:30 – 17:00 on the Friday afternoon preceding the Saturday sessions. Teachers work in teams, according to the grade they facilitate, to solve the questions that the learners are given on Saturday. The training sessions are conducted by the two Programme Directors: Mandisa Lebitso and Nina Scheepers.

The programme objectives for teachers and student teachers:

- Improve teachers' knowledge of maths content.
- Improve teachers' self-concepts with respect to the abilities to solve problems.
- Make teachers aware of problem-solving strategies.
- Make teachers aware of the value of developing problem-solving skills in learners.
- Make teachers aware that many problems can be solved in more than one way.
- Encourage teachers to use new and exciting methods in teaching maths.

The Saturday sessions are run at two local schools (Diamantveld H/S and Kevin Nkoane P/S) from 08:00 – 16:00. It is most inspiring to witness so many learners who are passionate about mathematics and who are willing to spend a whole Saturday engaged in maths problem-solving. No formal teaching takes place. Teachers serve as facilitators. The learners spend the most part of the day working in groups to find solutions to the maths problems and reporting their methods back to the class.

Through this process the learners:

- develop a willingness to try problems and improve their perseverance when solving problems.
- improve their self-concepts with respect to their abilities to solve problems.
- acquire problem-solving strategies.
- become aware that many problems can be solved in more than one way.
- improve their abilities to get more correct answers to problems.

After lunch the learners write a test which is marked immediately by the teachers. The results are recorded as a means of evaluation and for reporting to sponsors.

The MSLA Grade 9 learners' November school maths results have been compared with their maths problem-solving results revealing much higher results at school.

The need to address learners' maths and science problem-solving skills was raised in the article: MATHS AND SCIENCE PUPILS 'SET UP FOR DISAPPOINTMENT', by [K. Seekoei](#) and [D. Macfarlane](#) (07 Jan 2011), in response to the 2010 matric results:

Diane Grayson, a professor of physics at the University of Pretoria, said that focusing merely on pass rates in these subjects was not enough. To enable more learners to become engineers, scientists and medical practitioners it was necessary to improve their problem-solving skills and conceptual understanding in the two subjects, she said.



Problem solving Grayson headed a study published last year showing “anecdotal evidence” from universities that students’ problem-solving skills and conceptual understanding were worse than they were in the past. (Mail & Guardian, 07 Jan 2011).

### **PEDI: ENGLISH OLYMPIAD**

The Pilot English Development Initiative (PEDI) is a joint project of the De Beers English Olympiad, SACEE and the Maths & Science Leadership Academy NPC and is sponsored by the De Beers Fund. It aims to include English FAL speakers, within the De Beers operational areas, in the English Olympiad, by adding an educational direct contact element to deepen the learning experience, and enhance learners’ chances of success. All MSLA Grade 9- 11 learners participate in this programme as the specific outcomes of the PEDI English Olympiad Programme are directly linked to the language needs in education:

- building learners’ confidence in the use of English
- facilitating entry to the mainstream English Olympiad exam
- facilitating entry to and success in tertiary study in any field/chosen career

### **MSLA CLUBS**

Friday afternoons at the MSLA Campus are abuzz with other activities. Learners are invited to join various clubs: chess, journalism, robotics and STEM Leadership.

The MSLA Journalism Club is a source of motivation for the learners to write their own articles and reports on the events at MSLA and on whatever is important to them. The learners recognize the value of being in this club as they are developing so many skills: IT; reporting; language; team work and growing a sense of creativity and self-esteem. Some of their articles are included in the MSLA annual report to sponsors each year.

In today's global society, excellence in Science, Technology, Engineering, and Mathematics (STEM) is essential in contributing to building a workforce that will grow a healthy economy. The STEM Leadership Club is all about making the learners aware of this importance and creating a passion in them to serve as STEM ambassadors in the wider community. Last year they represented the organisation at the Bloodhound SSC exhibition at the Diamond Pavilion shopping mall during National Science Week and enjoyed interacting with the general public and encouraging people from all walks of life to become more aware of the importance of STEM in our modern world.

## **ESKOM EXPO FOR YOUNG SCIENTISTS COMPETITION**

When learners design, plan, carry out, and publicly exhibit a project of genuine value (to themselves, to the community, or to a teacher), it has a transformative effect on their perception of themselves, their relationship to learning, and their sense of their place in the world around them. It is also the best way to develop the diverse portfolio of skills that are increasingly in demand from universities and employers.

Each year the 190 MSLA Grade 9 – 11 learners are expected to enter a maths or science research project for the Eskom Expo for Young Scientists Competition.

The learners have access to the internet at the MSLA campus and they are provided with basic support and guidance in the early stages of compiling their projects.

During National Science Week, the MSLA learners have to present their projects to a panel of judges and the best projects are recommended for entry to the Provincial Eskom Expo Competition.

The response from the learners has been incredible and their efforts have been rewarded with a large number of learners receiving medals at provincial level and each year a number proceeding to the International Science Fair in Gauteng where they have done exceptionally well.

## **CAREER GUIDANCE**

The MSLA computer lab is a great asset to the organisation. It is the engine room that links us to the world wirelessly 24 /7 and enables the organisation to function professionally. It also plays a vital role in the development of our youth and in their preparation for life after school. The majority of our learners do not have access to computers or to the internet at their schools or at home. The matrics are able to apply to universities online and to further their own research.

The ultimate aim of these intervention programmes is to grow the pool of students who are well equipped to study further at tertiary institutions in maths and science related careers needed to grow South Africa's economy.

MSLA offers the learners support in career guidance, an area that is often neglected at schools. The Grade 9 learners receive guidance on their Grade 10 subject choices during the Spring Academy. The Grade 11 learners spend a week in the winter holidays experiencing the real world of work. Local companies and government departments have been most generous in opening their doors. There are few opportunities in Kimberley but each year the learners get to job-shadow alongside professionals associated with 2 to 3 of their possible choices of study. The Grade 12 learners receive one-on-one support from the MSLA Guidance Counsellor, who also assists them with the completion of application forms to universities and in accessing bursary information.



Each year the MSLA Grade 11 and 12 learners get to attend Open Days at tertiary institutions in Bloemfontein: CUT and UFS. Tertiary institutions further afield visit the MSLA Campus to make presentations to the matrics.

### **DIAMOND NETWORK – “GROWING THE MADIBA LEGACY”**

MSLA former students, going back to our launch in 2006, who are currently studying at universities around South Africa, have initiated the “**DIAMOND NETWORK**” – a group that wishes to reach out to MSLA Grade 12 learners and provide as much support as possible in making the journey from school to university a smooth ride. The group runs a 3-hour workshop for our matrics each year on the last day of the Winter Academy where they assign themselves as mentors to the matrics.

### **“KIDS TEACHING KIDS” PROGRAMME**

An excellent way of demystifying mathematics is to get kids to teach other kids.

The MSLA learners are expected to make a difference in their school communities through the “Kids teaching Kids” Programme which began in 2008. In that year the Grade 10 learners were trained to be teachers in the fields of maths, science, English and career guidance. They formed teams and each day during National Science Week a different team ran workshops for different groups of Grade 9 learners from various schools in Kimberley. All sessions were presented using 21<sup>st</sup> century equipment: laptops, data projectors and PowerPoint! By the end of the week they had taught over 500 learners.

During the winter holidays of the same year they ran workshops for other Grade 10s and impacted on over 200 learners.

The major event of that year was the “080808 Maths Extravaganza”. A hall was hired in Galeshewe, along with furniture for 420 people. On 08 August 2008 the 60 MSLA Grade 10 learners ran a day’s workshop for 360 other Grade 10s from 8 schools in Kimberley, together with their maths teachers. All participants received stationery, a CASIO calculator and good things to eat. Another great way of demystifying maths is to include stage smoke and a DJ. What an awesome maths experience! The day started at 08:00 but nobody wanted to stop doing maths at 15:00 and it was on a Friday!

In 2009 the Grade 10 group went to teach learners at 5 different schools in the Postmasburg area, 200 kms away from Kimberley, during the winter holidays. This meant that the day started at 05:00 in the cold and dark but nobody wanted to stay behind. The love for maths and science was the driving factor!

Since then the Kids teaching Kids Programme has become less formalised and all Grade 9 – 12 MSLA learners have to start their own programmes at their schools as a way of community service. School principals verify that this is actually happening.

It is hoped that through the Kids teaching Kids Programme:

- learners will become more responsible for their own learning
- the process of demystifying mathematics will occur
- more learners will believe they can do maths and science
- better results will be achieved
- the seeds for growing future maths and science teachers will be planted.

### **NATIONAL SCIENCE WEEK**

During National Science Week the organisation runs special maths & science programmes targeting learners, parents and the public at large. The programmes include hands-on workshops for 600 Grade 6 learners and 40 teachers, a “Kids teaching Parents” evening, MSLA Radio Talk Show on Radio Teemaneng, SET Career presentations for the 270 Grade 9 – 12 MSLA learners and a maths & science exhibition at the Diamond Pavilion shopping mall.

The MSLA Radio Talk Show is an amazing opportunity for getting learners to inspire other learners about:

- national and global challenges
- the need for young South Africans to take up maths and science-related careers
- skills needed in the 21<sup>st</sup> century
- various other relevant issues

The “Kids teaching Parents” evening is also a novel way of creating excitement about maths in the home. The session on kids teaching parents how to use the CASIO scientific calculator provides opportunity to make maths fun.

### **MSLA MATRIC RESULTS**

The first group of learners that started in the MSLA Maths and Science Programme in 2007 completed matric in 2010. It was very exciting to have the top mathematics learner of the province from our MSLA ranks. Goodwill Tshekela, a very poor learner from Dr E P Lekhela H/S in Galeshewe, was the only learner in the Northern Cape to achieve 100% for mathematics and 96% for science.

In 2011 the top learners for maths and science in the province both participated in the MSLA programmes. Since then MSLA has had learners in the top 20 positions in the province and the quality of the mathematics and science results has been very good.



In evaluating the results one must take into account the socio economic background of the learners as indicated below:

BOTH PARENTS	UNEMPLOYED	LIVE ELSEWHERE	DECEASED	UNKNOWN
	18	18	9	0
FATHER	UNEMPLOYED	LIVES ELSEWHERE	DECEASED	UNKNOWN
	18	52	37	32
MOTHER	UNEMPLOYED	LIVES ELSEWHERE	DECEASED	UNKNOWN
	71	32	10	2

TABLE 1: PARENTS’ SITUATION OF ALL GRADE 9 – 12 LEARNERS

LIVES WITH: AUNT/ SISTER /GRANNY / GRANDFATHER	40
LIVES IN SHANTIES/SHACKS	13
NO PLACE TO STUDY AT HOME	64
NO ELECTRICITY AT HOME	13

TABLE 2: HOME CONDITIONS OF ALL GRADE 9 – 12 LEARNERS

SUBJECT	RANGE %	NUMBER	GROUP%
MATHEMATICS	90 - 100	4	7.14
	80 - 89	10	17.86
	70 - 79	9	16.07
	60 - 69	14	25.00
	50 - 59	14	25.00
	40 - 49	3	5.36
	30 - 39	2	3.57
TOTAL		56	100

**QUALITY OF RESULTS:**  
 25% achieved above 80%  
 41.07% achieved above 70%  
 66.07% achieved above 60%  
 91.07% achieved above 50%

TABLE 3: QUALITY OF MSLA MATHEMATICS RESULTS: 2013 GRADE 12

SUBJECT	RANGE %	NUMBER	GROUP %
PHYSICAL SCIENCE	90 - 100	1	1.79
	80 - 89	4	7.14
	70 - 79	13	23.21
	60 - 69	10	17.86
	50 - 59	13	23.21
	40 - 49	9	16.07
	30 - 39	6	10.71
TOTAL		56	100

**QUALITY OF RESULTS:**  
 8.93% achieved above 80%  
 32.14% achieved above 70%  
 50% achieved above 60%  
 73.21% achieved above 50%

**TABLE 4: QUALITY OF MSLA PHYSICAL SCIENCES RESULTS:**  
**2013 GRADE 12**

It is pleasing to note that of the 56 matrics in 2013, 48 are studying further in 2014 at universities or universities of technology. Of these, 42 are first generation at tertiary institutions.

**MSLA – THE OPPORTUNITY ISLAND VISION**

De Beers has donated 1.9ha of land in Kimberley for MSLA to build its dream venue: OPPORTUNITY ISLAND. This concept is unique and innovative and will certainly address many of the challenges facing education, especially maths and science, in the Northern Cape Province.

The vision is to create a 21<sup>st</sup> century educational environment for 21<sup>st</sup> century teaching & learning, that will:

- break the cycle of poor education,
- inspire a passion for maths & science,
- provide opportunities for developing innovation and creativity
- set new standards & instill values,
- lead to future scientists, engineers & visionary leaders of calibre, so as to grow the economy of the Northern Cape and reduce the level of poverty

There will be OPPORTUNITIES...

- for all stakeholders, with a vested interest in education, to make a collective difference in breaking the cycle of poor education
- for all partners, to have a share in creating a WORLD-CLASS 21<sup>ST</sup> CENTURY teaching and learning environment aimed at empowering top quality maths and science teachers and learners



- for professionals, to engage with learners at school level in view of motivating and inspiring the youth to follow in their footsteps
- for the wider community, to experience a wide range of exciting maths & science related activities and become a society that values STEM
- for the unemployed, especially the youth, to receive training and develop skills

OPPORTUNITY ISLAND will become an ISLAND OF OPPORTUNITIES FOR ALL that will inspire a new generation of visionary leaders, who have the knowledge, values and determination TO BUILD THE MAINLAND WE WANT!

Kumba Iron Ore views the Maths & Science Leadership Academy NPC on Opportunity Island as being the solution to their planned mega project for the Northern Cape Province and has already sponsored the cost of the feasibility study that has been carried out by PricewaterhouseCoopers.

The plan is to establish a full-time Maths and Science Leadership Academy on the island for Grade 8 —12 learners with potential from across the province, especially from the rural areas, who will receive quality education in a 21st century learning environment.

Schools from all the corners of the Northern Cape Province that lack resources and good maths and science teachers will receive assistance from Opportunity Island through the use of 21<sup>st</sup> century technology. In this way, teachers will be empowered, more learners will have the opportunity to study maths and science and our youth will enjoy the bright futures they deserve.

OPPORTUNITY ISLAND will continue to offer:

- Afternoon classes & holiday programmes for learners in Kimberley
- Workshops & empowerment sessions for teachers
- Maths & science problem-solving programmes
- Skills development programmes

GXY Architects have already completed the plans for the building which is to be totally GREEN!

The lower level will include:

- auditorium (300 seats)
- 2 computer labs
- science lab
- technology lab
- 5 multi-purpose areas (with cooking facilities)
- multi-purpose hall (1200 seats)
- cooking school, veggie garden and greenhouse
- music room
- games area
- rock-climbing wall

- radio station
- STEM exhibitions area
- tuck-shop

The upper level will include:

- 10 classrooms
- boardroom
- staffroom
- admin offices
- media centre
- careers centre

There will be A WORLD OF OPPORTUNITIES ... (to name but a few)

- Teacher training (internships)
- Focus weeks
- Career guidance
- Clubs: press, chess, debating, music, etc.
- ICT training
- Competitions & Science Expos
- Kids radio station
- Annual Maths & Science Conference (For kids — by kids)
- Job creation
- Sports & cultural events
- A “water plant” - promoting water-related careers

The establishment of **OPPORTUNITY ISLAND** in Kimberley will ensure more students gain entrance to the new Northern Cape University, thus assisting in breaking the cycle of poverty and ensuring economic growth.

The Northern Cape Provincial Growth & Development Strategy acknowledges the vital importance of developing human and social capital: “Creating opportunities for life-long learning; improving the skills of the labour force to increase productivity and increasing access to knowledge and information”.

MSLA, in its quest to enhance mathematics, science and technology competencies of a wide range of target groups, has the potential to play a key role in assisting the Northern Cape Province in its strategy to build a stronger economy and reduce poverty.

**OPPORTUNITY ISLAND** will be a source of hope and inspiration – a benchmark for setting standards – a learning organization providing opportunities for the development of human capital and life-long learning – an island of quality education.



## CONCLUDING COMMENTS

Things cannot go on as they are! Demystifying mathematics is everyone's business. So let's:

- *rekindle the fire of passion for what we do*
- *strive to become world class maths teachers*
- *take ownership where we have failed*
- *change what can be changed*
- *ensure a bright future for our youth*
- *acknowledge the importance of our role as mathematics teachers in growing the economy and reducing poverty levels*

*Together, let's start building the world we want - let's keep our world alive!*

## REFERENCES

- Campbell, G. and Prew, M. (2014). *Behind the matric results: the story of maths and science*. Mail & Guardian, 07 Jan 2014.
- DBE. (2013). NSC Exam Reports – Technical Report: Department of Basic Education
- Jansen, L. (2013). *SA's shocking literacy stats*. IOL News, 22 October 2013.
- McCarthy, J. and Oliphant, R. (2013). *Mathematics outcomes in South African schools. What are the facts? What should be done?* CDE Insight, October 2013.
- Prew, M. (2013). *Our maths mind-set doesn't add up*. Mail & Guardian, 05 July 2013.
- Schleicher, A.** (2010). *The case for 21st-century learning*. OECD.
- SEEKOEI, K. AND MACFARLANE, D. (2011) *MATHS AND SCIENCE PUPILS 'SET UP FOR DISAPPOINTMENT'*. MAIL & GUARDIAN, 07 JAN 2011.
- Snyman, J. (2013). *Not adding up: too few maths teachers to satisfy demand*. SAIRR Press release, 17 April 2

## SO, WHAT DO OUR TEACHERS KNOW?

Vimolan Mudaly

University of KwaZulu-Natal

### *Extended abstract*

Globalisation has created many challenges for teacher training institutions. Knowledge is available almost instantaneously using media that is becoming easily accessible to people everywhere. Having knowledge virtually at the fingertips does not solve the problems of pedagogy in classrooms. We still need to train teachers who are knowledgeable in mathematics and its applications as well as pedagogical skills related to mathematics teaching. Knowing either the mathematical content well or having good pedagogical knowledge does not guarantee successful learning. This paper describes the different types of knowledge a mathematics teacher ought to possess and claims that a mathematics teacher must be well-practiced in the subject content matter. Well-practiced will imply a teacher with a well-connected and well-understood conceptual knowledge.

Fullan (1993) stated that “teacher education has the honor of being simultaneously the worst problem and the best solution in education”. It is a problem in mathematics because we are not producing sufficiently good teachers and yet it is within the ambits of our own curricula that we can make a difference. There is little doubt that the onus is on teacher training institutions to reconsider how they can prepare effective future teachers to prepare learners well as they enter a rapidly changing world.

All of this is particularly important for South African mathematics classrooms. South African students’ mathematics results are generally poor. Often, we seek to blame students and their previous teachers for their current performances in tests and examinations but do our teachers really know what they are teaching? This paper looks at the discourses in mathematics from the context of meaning making and it urges teachers to become reflexive practitioners. The paper also looks at some research conducted within the field of mathematics education.

There are many indications that as mathematics educators we face a barrage of difficulties and, currently, there seems to be no magic wand to rectify the situation. Mcgregor (2009), in a report in the University World News, stated that “higher education is facing a very serious problem in respect of the mathematics knowledge and manifest ability of its entering classes”. In 2009, almost 13 000 students from about 5 higher educational institutions wrote a benchmark test in their first year. The results were indeed shocking. Most frightening of all were the mathematics results. Only 7% of students were found to be proficient in the tests, which measured the skills needed to study first-year mathematics.



Some 73% had intermediate skills and would need assistance to pass, while 20% had basic skills (Mcgregor, 2009) and would need long term support. So what changed in our teaching after this report was released? Whether schools prepare students sufficiently well enough or not, what have we, as teacher educators, done to make a tangible improvement to the results? In particular, what have mathematics teacher educators done to produce mathematics teachers that are capable of turning the tide against our poor school performances?

This paper looks at empirical evidence obtained from final year Bachelor of Education and Post Graduate Certificate in Education students' knowledge. The evidence obtained is showing that all teacher educators need to be concerned and, radical improvements must be made in order to reverse the trend of poor performances in schools. Simple concepts were tested in teacher education classes and the responses scrutinized and the analyses obtained show that our pre-service teachers lack a deep conceptual understanding of simple aspects in school mathematics.

## MATHEMATICAL STRUCTURES, AND TIGHT CONNECTIONS BETWEEN THEM

Ingrid Rewitzky

*Department of Mathematical Sciences, Stellenbosch University*

A central notion arising in twentieth century mathematics is that of a mathematical structure. Through identifying the essential features of some entity (such as number, space, symmetry, network) or activity (such as counting, collecting, rearranging, computing, arguing) arising in an application domain (such as engineering, physics, computer science, philosophy) and understanding it as deeply and completely as possible, we move from thinking about specific entities or activities to thinking about more abstract entities or activities. In this way the notion of a mathematical structure evolves from entities and activities of an application domain, which may then find applications in another application domain.

While there is agreement that everything is a mathematical structure, there is a diversity of mathematical opinions of an appropriate formalism of a mathematical structure.

David Hilbert, in "Uber das Unendliche, *Mathematische Annalen* (95): 161-190 (1926) said:

"Noone will drive us from the paradise which Cantor created for us.". Some 'attacked' set theory. For example, Wittgenstein replied

"If one person can see it as a paradise of mathematicians, why should not another see it as a joke?".

Others like Mac Lane felt that this

"is a mistakenly one-sided view of mathematics"

and his letter *Mathematical Models: A Sketch for the Philosophy of Mathematics*, in *The American Mathematical Monthly*, 88 (7) (1981) p 462-472, may be interpreted as saying that

"In spite of the fundamental achievements of set theory, the perfect paradise is still to be found."

From such different opinions about how to formalize the notion of a mathematical structure, different fields of mathematics (including Algebra, Logic, Topology, Category Theory) have evolved.



Typical questions then are whether two structures (not necessarily from the same field of mathematics) are in essence the same, or really different, or whether one structure might be obtained from the other via certain natural constructions.

Establishing such tight connections contributes substantially to each field of mathematics by solving some of its shortcomings (for example, solving a problem in a given mathematical field by incorporating techniques from another mathematical field), and by providing deeper understanding of the fundamental differences between fields of mathematics, which sometimes may even lead to their unification. Moreover, different mathematical structures may provide different perspectives of the entities and activities of a given application domain, and often the interplay of these different perspectives illuminate the subject matter.

Our focus in this talk will be on exploring mathematical structures and the connections between different mathematical structures, together with their applications within and outside mathematics,

## CONVERSIONS IN THE METRIC SYSTEM: REFLECTING ON THE GET MATHEMATICS CURRICULUM

Stanley A. Adendorff

Cape Peninsula University of Technology

*In this paper I argue that a fragmented approach to the teaching of the SI (Standard International) units – as dealt with in the National Curriculum Statement/ Curriculum and Assessment Policy Statement - Intermediate and Senior Phase Mathematics as part of General Education and Training – (GET) - when compared to the decimal system (with base 10), may interfere with understanding and conceptualisation, and with solving problems involving conversion for both learners and teachers. I also argue that isolating and fragmenting SI measurement units, complicate processes of accommodation and assimilation as advocated by Piaget. In terms of the Gestalt learning theory, fragmentation (in this context) prevents learners from seeing the greater meaning of the relationship(s) that exist among the different SI units. It thus appears, as if the curriculum writers are in favour of a more instrumental approach to parts of the curriculum as opposed to relational, which essentially also focus on the underlying mathematical relationships to enhance sense-making.*

### INTRODUCTION

If anybody in the street were to be asked to express an opinion on the position, role or importance of measurement in the Intermediate and Senior Phase mathematics curriculum (as currently used within the National Curriculum Statement in South Africa), such a person would most probably recognize and appreciate its value in terms of everyday requirements; economically, technologically, and socially. Yet, the actual time allotted to measurement in the mathematics classroom, let alone whether this is time spent effectively and efficiently, needs to be considered.

This opinion paper critiques the fragmented and instrumental approach to metric conversion in the NCS for the GET (which refers to the General Education and Training band, from Grade R to grade 9). I argue that fragmentation of the SI (Standard International) units, especially, when compared to the decimal system (with base 10), interferes with understanding and effectively dealing with problems involving conversion for both learners and teachers. I also argue that isolating and fragmenting SI measurement units, complicate processes of accommodation and assimilation as advocated by Piaget (1968). Furthermore, that in terms of the Gestalt learning theory fragmentation prevents learners from seeing the “greater meaning” (Dabba, 1999, p. 1) of the relationships that exist (Bragg & Outhred, 2004) between the different SI units. Thirdly, instrumental learning as opposed to relational learning, reinforces rote learning and the use of rules without understanding the underlying mathematical concepts (Skemp, 2006, p. 88).

My experiences as an in-service trainer of teachers at CTLI (Cape Teaching and Leadership Institute, Western Cape Education Department) since 2005 had given me the impression that measurement or mensuration as part of the intermediate and senior phase mathematics seemed to be problematic. Several years of facilitation at schools in the Western Cape, and observing learners doing the CTA (Common Task for Assessment – an externally set assessment written towards the end of the Senior Phase, that is, grade 9) showed that most of them could not measure accurately, and had limited knowledge about the relationships between units of length for instance. The CTLI is the official training arm of the Western Cape Education Department and runs in-service programs for Foundation, Intermediate and Senior Phase teachers. These teachers attend four-week courses (as fortnightly sessions) on a full-time basis, but only if a substitute is appointed for the duration of the course.

Instead of allowing learners to be confronted by the bigger whole (from kilo ( $10^3$ ) to milli ( $10^{-3}$ ), curriculum developers seem to prejudge Intermediate and Senior Phase learners' cognitive abilities by anticipating that they would not be mentally ready to adequately comprehend or meaningfully deal with the concepts mentioned. These concepts refer to the meaning of and relationships between units of length, units of capacity or units of mass.

### **Rationale for pursuing this topic**

Based on my experiences in mathematics classrooms while observing lessons, the effectiveness and efficiency of methodologies dealing with measurement, compelled me to look at the situation analytically. According to (Kamii and Clark, 1997, p. 120) “typical instruction treats measurement as a mere empirical procedure rather than ... requiring reasoning”. Consequently, Osborne's (1980, p. 54-68) critical analysis of the role of measurement in the mathematics classroom deserves to be looked at closely.

Osborne (1980, p. 55) emphasises the need for effective organization of teaching strategies to focus primarily on the “fundamental characteristics of measurement” that forms the basis of any measurement system. He believes this would allow the learner “to eventually transfer what was learned from one system of measurement to another”. At the same time, however, measurement ideas and concepts should be applied to facilitate the development of numerical skills and understanding of related concepts. Teaching programs and lessons on measurement are not effectively geared towards acquiring the crucial concepts of measurement (Hiebert, 1984; Battista, 2006). Osborne (1980, p. 56) maintains that learning activities related to measurement generally do not “help learners form a sufficiently powerful ideation structure to enable them to deal successfully with new learning and problem solving involving measurement”. Kamii and Clark (1997, p. 121) concur that learners should be encouraged through teaching “to think hard and to modify their thinking, rather than teach empirical procedures that do not take their thinking into account”. Similarly Battista (2006, p. 140) talks of a “major challenge in teaching to [help] students make genuine sense of mathematical ideas”.

Teaching, involving the use of different measurement tools, generally tends to be very limited and superficial. In this regard Osborne (1980, p. 56) states that “without a sense of the nature of the relationship between these tools and what is being measured, little insight into the scientific and mathematical process is obtained”. This can also be ascribed to “insufficient experience with preliminary concepts” and insufficient “opportunity to tie together a base of geometric understanding with actual number concepts of measurement”. This is possibly manifested in learners’ inability to manipulate numbers, and to meaningfully convert from one unit of measurement to another, be it related to mass, capacity or length.

### Theories that inform this critique

My stance with respect to this approach to the learning and teaching of the topic under discussion is adequately supported by one of the classical learning theories, Gestalt, as to how learners order and rearrange concepts mentally to make better sense. Furthermore, Piaget’s ideas and findings about learners’ use of previous knowledge to enhance assimilation and conceptualisation of new concepts are revisited. Thirdly, Skemp (2006) differentiates between instrumental and relational understanding of mathematical concepts which highlights the difference between rote learning and learning for understanding by focusing on underlying relationships.

Gestalt theorists were fascinated by the way the human mind “perceives wholes out of incomplete elements” (Skaalid, 1999, p. 1). According to the Gestaltists, things or objects are affected by where they are, that is, their position in space and by what surrounds them. Consequently, these objects are more appropriately described as much “more than the sum of their parts” (Behrens, 1984, p. 49). This implies that context was considered to be very important as far as perception is concerned.

Of the six laws of perception advanced by Gestaltists, the one that deals with ‘closure’, seems to be important here. This law states that the human mind, in the case of incomplete shapes, experience[s] these shapes as incomplete and compels “the learner to want to discover what’s missing, rather than concentrating on the given instruction” (Dabbagh, 1999, p. 3). Here I want to refer to the gap that is created between, for example, meter and centimeter, by the omission of decimeter in the curriculum with respect to the SI system. Another principle, namely that of ‘similarity’ states that “things which share visual characteristics such as shape, size, color, texture, value or orientation” (Skaalid, 1999, p. 3) will be perceived as belonging together. In this case it is obvious, for example, that the 1000 in the decimal number system corresponds to kilometre as 1000 metres in the SI system.

The Gestalt theorists were mainly interested in “the entirety (the whole) of the problem or experience. To Wertheimer [a Gestaltist], truth was determined by the entire structure of experience rather than by individual sensation or perceptions.” (Dabbagh, 1999, p. 5). In this regard I think of “broken up” bits of knowledge such as  $1000\text{ mm} = 1\text{ m}$ ,  $100\text{ cm} = 1\text{ m}$ , etc. that learners are expected to memorize and apply in order to do conversions.



Piaget's ideas of assimilation and accommodation were mentioned earlier on. At this stage I briefly want to touch on the meaning of these terms together with his use of the concepts 'schema' and 'equilibrium'. Assimilation takes place when the learner perceives new phenomena or occurrences in terms of existing schemas (mental building blocks). Accommodation, which "refers to the process of changing internal mental structures to provide consistency with external reality", "occurs when existing schemas or operations must be modified or new schemas are created to account for a new experience" (Bhattacharya and Han, 2001, p. 2). The internal process of a learner making sense of external occurrences according to his or her internal experiences in order to achieve a balance between assimilation and accommodation is referred to as 'equilibrium'.

It is my contention that by allowing learners to work with the SI units in relation to the decimal number system accommodation and assimilation are facilitated and that a state of equilibrium is reached sooner. This approach emphasises and highlights the link between, and integration of the two systems

The issue of fragmentation or compartmentalisation also links up with Skemp's (2006) differentiation between instrumental and relational understanding of teaching mathematics. Instrumental understanding methods relate to the rote learning of rules, laws, procedures and algorithms, that is, learners overwhelmingly depend on guidance for learning new methods or techniques through knowledge transmission. Relational understanding methods relate to understanding relationships and connections between phenomena, concepts or mathematical ideas, allowing learners insight and knowledge as to what to do and why. Skemp (2006, p. 95) maintains that "learning relational mathematics consists of building up a conceptual structure (schema) from which its possessor can ... produce an unlimited number of plans for getting from any starting point within his schema to any finishing point".

### **Problems relating to conversions**

By isolating or compartmentalising certain SI units, the curriculum developers may be denying the learner the chance to explore, make better sense of and get a better grip on the relationships that exist among the different SI (metric) units and the decimal (denary) system. Coherence in a sense is thus forfeited. At the same time one wonders what the reasons or rationale might be for approaching measurement in this manner. Is it related to the notion that "relational understanding would take too long to achieve" or that the relational understanding of this "particular topic is too difficult" (Skemp, 2006, p. 93)?

The problems that both learners and teachers encounter in this regard seem to be related to their sense of, "the nature of the relationship" (Osborne, 1980, p. 56) between different measuring units of length for example. Kamii (2006, p. 155) maintains that learners often did not grasp units of measurement which may relate to how estimation is taught and the kind of estimation activities used to develop measurement concepts.

For some reason the value of estimation to develop, “reinforce and elaborate the fundamental structural concepts of measurement” (Osborne, 1980, p. 55) may be under-valued.

The value of estimation is generally considered to be crucial for concept development. Osborne (1980, p. 61) maintains that “estimation skills require constant and frequent practice or they evaporate”. He also states that although “any estimate is correct in an absolute sense”, “the premium must be on which estimate is better”. Activities that allow learners the opportunity to compare their estimates with actual accurate measurement should be built into the process of concept development. These estimation activities should be applied with respect to length, mass, volume, temperature, etc. This would ensure that estimation is used in a variety of situations, thereby enhancing learners’ understanding of its usefulness (Osborne, 1980, p. 66).

Asking learners to show or indicate different lengths by using their fingers, arms or feet reflected that they generally had a limited sense of the meaning of length or distance. A typical activity (see Table 1 below) for estimating distance or mass or capacity / volume estimation exercises could look as follows:

object	Estimation of length or capacity or mass	actual measurement	difference
--------	--	--------------------	------------

Table 1: Estimation

### A practical way of counteracting difficulties with conversion

Some pertinent questions to pose are the following: What are the differences between our number system and the SI system? What are the similarities between our decimal number system and the SI system? Why not use these similarities as a foundation from which to expand understanding of the different measuring units? Integration of concepts would facilitate understanding (Kheong, 1997), as well as the use constructs to enhance conceptualisation, and not fall back on compelling learners to memorise statements such as  $100\text{ cm} = 1\text{m}$ ,  $1000\text{ mm} = 1\text{m}$ , etc. without learners fully realising the meaning thereof.

Teaching conversion in context, that is seeing all parts in relation to one another, rather than working with numerical values in isolation, makes much more sense to learners. Learners have difficulty making the connection when dealing with concepts out of context. What does it mean to convert 1 m to cm? Would a grade 4, 5 or 6 learner be able to say that the 1 m is cut up into a 100 little lengths of 1cm each? It would make more sense to do this in terms of a physical object such as a metre long plank or piece of rope or string. What does it mean to convert 0,8 g to mg? Does it mean to the learner, dividing the 0,8g up into 1000 small quantities, in this case milligrams (mg)? Does it mean how many full milligram quantities can you get out of 0,8g? Or does it mean magically changing the quantity to something a thousand times ‘bigger’ or ‘smaller’, referred to as “unit iteration (Kamii & Clark, 1997, p. 120)?



In the NCS (DBE, 2011, p. 25) the specific content on length is indicated as follows:

Grade 4: Conversions include converting between millimetres (mm), and centimetres (cm); centimetres (cm) and metres (m); metres (m) and kilometres (km). Conversions limited to whole numbers and common fractions

Grade 5: Conversions include converting between any of the following units: millimetres (mm); centimetres (cm); metres (m) and kilometres (km)

Conversions limited to whole numbers and common fractions

Grade 6: Conversions include converting between any of the following units: millimetres (mm); centimetres (cm); metres (m); kilometres (km). Conversions should include common fraction and decimal fractions to 2 decimal places

The “list above” only contains the following: “length using millimetres (mm), centimetres (cm), metres (m) and kilometres (km). Did the curriculum developers anticipate that to include the other units of length would hamper conceptualisation, or not have the “logico-mathematical knowledge” to make mental relationships (Kamii, 2006, p. 158)? Why for instance is ‘decimetre’ not mentioned in the curriculum statement, taking into consideration its close relationship with volume ( $1\text{dm}^3 = 1$  litre)? Furthermore, the term or concept *hectometre* is not mentioned anywhere in the NCS or RNCS (Revised National Curriculum Statement), yet calculations involving *hectare* appeared in one of the past CTA’s (Common Task for Assessment, General Education and Training).

Before proceeding to the use of the conversion table it needs to be emphasised that to deal with conversions successfully also implies an understanding of the nature of the SI measurement system. According to Osborne (1980, p. 56) such an understanding of the characteristics of the metric system means that the learner “possesses an ideational structure that can serve as a base both for problem solving and for measurement ideas”. This structure according to Osborne (1980, p. 56-57) includes the following:

1. Number assignment: To measure an object is to assign a number to an attribute or a state of the object...
2. Comparison: If object A is “contained by” Object B, then the measure of object A is less than the measure of object B...
3. Congruence: If object A and object B are congruent, then the numbers that are their measures are the same.
4. Units with iteration: There is a special object or state in any measurement system to which the number one is assigned [such as metre to measure distance in the SI system]. The object with measure one is the base for identifying the functional rules that determine the number assignment for any object or state...
5. “Additivity...” (addition of two or more lengths).

In this discussion I make use of the ‘conversion’ table below as an attempt to explain a particular approach of dealing with conversion of one SI unit to another. The table should be seen and used as a tool, especially with learners who experience difficulty converting from one unit of measurement to another. It is an auxiliary tool to help learners see a particular value in context and in relation to another, for example 0,002 km as opposed to 2 meters. It needs to be emphasised that all values used in the table below should be read in relation to the meter as unit of measurement. A ‘typical’ conversion table could look like this:

↑	Thousands 1000 (10 <sup>3</sup> )	Hundreds 100 (10 <sup>2</sup> )	Tens 10 (10 <sup>1</sup> )	Units 1 (10 <sup>0</sup> )	Tenths 1/10 (10 <sup>-1</sup> )	Hundredths 1/100 (10 <sup>-2</sup> )	Thousandths 1/1000 (10 <sup>-3</sup> )	↓
LARGER QUANTITIES OR DISTANCES mega meter	kilo- metre	hecto- metre	deca- metre	metre	deci- metre	centi- metre	milli- metre	SMALLER QUANTITIES OR DISTANCES nanometre
	kilo- gram	hecto- gram	deca- gram	gram	deci- gram	centi- gram	milli- gram	
	kilo- litre	hecto- litre	deca- litre	litre	deci- litre	centi- litre	milli- litre	
				5	0	0cm		
				5	0	0	0mm	
	0,	0	0	5km				
				5	0dm			
	7km							
7	0	0	0 m					
7	0	0	0	0	0cm			

Table 2: Distance/mass/volume

Using the above-mentioned (Table 2) makes it quite easy to convert 5m for example to centimetres, millimetres or to write it as a decimal fraction of a kilometre. From Table 2 it is evidently much clearer to follow what actually happens. It is easy to see the correspondence between the decimal (denary) number system and the SI system. It is evident that the bigger picture is much more than the mere sum of the constituent parts. By leaving out or ignoring for instance the decimetre, hectometre or decametre as units of measurement gaps or voids are created that may cause problems for learners. Starting with the Gestalt (the whole), and allowing learners to perceive the interrelationships and inter-connectedness of the different measuring units, I maintain would be experienced as much more meaningful and enhance assimilation.



From the table, using metre as point of reference, it is also easy to observe that ‘kilo’ means ‘thousand’, ‘hecto’ means ‘hundred’, ‘deci’ means ‘tenth’, ‘centi’ means ‘hundredth’ and ‘milli’ means ‘thousandth’ in terms of for instance the metre as SI unit to measure distance. This means that the position of each is important and indicates its value in relation to others. The table could be altered to include only information that relates to capacity or length or mass when dealing with specifics for practical reasons. The table as it appears above could thus be adapted according to the level of the learner.

Does the curriculum really advocate fragmentation and compartmentalisation of mathematics content by dealing with SI units in the following manner: 1 metre = 1000 mm or 1mm = 0,001 m, etc.? One gets the sense that the curriculum in this regard advocates an instrumental approach to learning mathematical concepts. Learners are supposed to learn and know this by heart – and not necessarily with insight and understanding. By isolating the units and dealing with them as separate entities makes it more difficult for learners to comprehend. Teaching focuses more on “establishing familiarity with and use of the metric units, than in using learning about metric measurement to reveal the nature of measurement” (Osborne, 1980, p. 56). Consequently, learners may end up “with several particulate bits of discrete knowledge about measurement rather than developing a feel for and an understanding about the nature of measurement as a system”.

### **Simultaneously dealing with decimal fractions**

A contentious issue is that of ‘shifting’ the comma when converting from one unit to the other. Does the comma shift? Or is it just a convenient way of explaining what happens when converting? Dealing with conversions as suggested also creates the opportunity to address decimal fractions in a much more meaningful way - that is, in context and therefore much more realistic. Telling learners to divide a 0,8 metre length of wood into 5 equal pieces, makes much more sense than merely telling learners to divide 0,8 by 5.

From the Table 2 one could easily determine what decimal fraction 5 metres is of a kilometre, or of a hectometre. Similarly centimetres, decimetres or millimetres could be written as a fraction of a metre. Simple addition and subtraction manipulations could also be done using a table like this. Adding or subtracting quantities can be done using the conversion table (see Table 3). For example to add 45 g, 2 876 cg and 8 943mg could pose problems when learners try to add the values or quantities either by writing them underneath one another or next to one another in the same line. In this case the answer can either be written as 82,703 g or 8270,3 cg or 82703 mg or even as 0,082703 kg as the quantity “is being conserved” ( Hiebert, 1981, p. 208). This means that the teacher must give specific instructions and tell learners whether the answer should be given in grams, centigrams or milligrams, or kilograms. The following table can be used to accurately position the various quantities:

↑	Thousands 1000 (10 <sup>3</sup> )	Hundreds 100 (10 <sup>2</sup> )	Tens 10 (10 <sup>1</sup> )	Units 1 (10 <sup>0</sup> )	Tenths 1/10 (10 <sup>-1</sup> )	Hundredths 1/100 (10 <sup>-2</sup> )	Thousandths 1/1000 (10 <sup>-3</sup> )	→
LARGER QUANTITIES	kilo-gram	hecto-gram	deca-gram	gram	deci-gram	centi-gram	milli-gram	SMALLER QUANTITIES
			4	5g				
			2	8	7	6cg		
		+		8	9	4	3mg	
			8	2	7	0	3	

Table 3: Mass

### The role of actual measurement

The role of actual measurement, that is to physically explore the relationship between SI units, is essential before working with them theoretically as suggested with the unit table. None of this can occur if learners were not exposed to actually measuring and comparing the different units of length for example. They should practically count how many millimetres there are in a centimetre, centimetres in a decimetre, decimetres in a metre, etc.

Learners have to be taken outside to walk 10, or 100 or 1000 metres to help them conceptualize the magnitude of decametre, hectometre, etc. This very seldom happens inside the real classroom. This state of affairs was confirmed by a few hundred intermediate phase teachers who went through CTLI training during the course of 2005, 2007 and 2008. The teacher should thus create opportunities for contextualisation with SI units, as well as much greater integration with learning areas such as Geography and Natural Sciences.

### CONCLUDING COMMENTS

Based on my experience as facilitator and trainer, teachers tend to shy away from teaching topics or aspects of mathematics that they do not understand or that they have limited knowledge of. This is also true in the case of conversion. Many of the 200 teachers at the CTLI (2005) who attended my workshops on measurement, claimed that they had a much better understanding of conversion and that they would apply a practical approach (Hiebert, 1984) in their classrooms. It is interesting that many of them never heard of decimetre or hectometre before. They claimed that this holistic approach facilitated their understanding and assisted them in converting quantities quicker and more accurately.



It is important to realise that statements such as  $1000\text{ m} = 1\text{ km}$ ,  $100\text{ cm} = 1\text{ m}$ , etc. can still be learned by heart, but only once learners have a good idea of the bigger picture as discussed earlier. This would allow them to understand that 1000 metres is exactly the same 1 kilometre, since *kilo-* means *thousand*, for instance. A deliberate effort should thus be made to help learners understand and learn the meanings of all the Latin prefixes such as *kilo-*, *hecto-*, *deca-*, *deci-*, *centi-* and *milli-* in terms of the concept metre.

The use of Table 2 or variations thereof should be viewed as a tool to help learners see relationships and develop insight into what happens when converting from one unit of measurement to another. Its value is especially evident when introducing conversions. Once learners have gained insight, the use thereof may be discontinued or it can be used as a means of reference.

It is apparent that an instrumental approach to the learning of mathematics necessitates a teacher-centred approach of transmitting rules and procedures which is limiting and less adaptable to new tasks (Skemp, 2006; Bragg & Outhred, 2004). In contrast, teaching for relational understanding requires a more learner-centred approach by giving learners scope and allowing them opportunities to explore relationships, consequently fostering independent thinking, and enhancing understanding and remembering.

## REFERENCES

- Battista, MT. (2006). Understanding the development of students' thinking about length. *Teaching Children Mathematics*, October, 13(3), 140–147.
- Bhattacharya, K. and Han, S. (2001). *Piaget and Cognitive Development*. Retrieved March 04, 2006, from <http://www.coe.uga.edu/epitt/piaget.htm>.
- Behrens, R. (1984). *Design in the visual arts*. Englewood Cliffs, NJ: Prentice-Hall, Inc.
- Bragg, P. and Outhred, L. (2004). A measure of rules – the importance of units in a measure. *Proceedings of the 28<sup>th</sup> Conference of the International Group for the Psychology of Mathematics education*, 2, 159–166.
- Dabbagh, N. (1999). *Gestalt and instructional design*. Retrieved January 16, 2006, from <http://chd.gse.gmu.edu/immersion/knowledgebase/strategies/ cognitivism/ gestalt/gestalt.htm>.
- Fong Ho Kheong, FH. (1997). An integrated approach for teaching mathematical problem solving. *The Mathematics Educator*, 1 (1), 36-44.
- Hierbert, J. (1981). Cognitive development and learning linear measurement. *Journal for Research in Mathematics Education*, 12(3), 197-211.
- Hiebert, J. (1984). Why do some children have trouble learning measurement concepts? *Arithmetic Teacher*, March, 3(7), 19-24.
- Kamii, C. (2006). Measurement of length: how can we teach better? *Teaching Children Mathematics*, October, 154-158.
- Kamii, C. and Clark, FB. (1997). Measurement of length: The need for a better approach to Teaching. *School Science and Mathematics*, 97(3), 116-121.
- Osborne, A. (1980). Measurement: How Much? In M.M. Lindquist (ed.), *Selected issues in Mathematics Education*. Chicago: McCutchan Publishers

- Piaget, J. (1968). *On the development of memory and identity*. Barre, MA: Clark University Press with Barre Publishers.
- Skaalid, B. (1999). *Gestalt principles of perception*. Retrieved March 06, 2006 from <http://www.usask.ca/education/coursework/skaalid/theory/gestalt/gestalt.htm>.
- Skemp, R.R. (2006). Relational understanding and instrumental understanding. *Mathematics Teacher in the Middle School*, 12(2), 88 – 95.
- South Africa. Department of Education. (2012). *National Curriculum Statement Grades R-9 (Schools) Policy*. Pretoria: Government Printer.



## EXPLORING LEARNING TRAJECTORIES IN ALGEBRA: INFORMING THE TEACHING OF FUNCTIONS

Robyn Clark

### *Extended abstract*

When teaching functions, mathematics teachers often know the outcomes that their learners are expected to achieve, but do not necessarily know the best way in which to teach to ensure that their learners achieve these outcomes. Learners are not able to grasp a concept in its entirety after seeing it only once. The pathway to understanding is a long one, and it takes time to learn a concept.

Learning trajectories chart the paths of learning, instead of just looking at the outcome of learning. The South African curriculum states its learning objectives in terms of outcomes, but does not say how these outcomes should be reached. It seems that many school curriculums are similar, and hence the path of learning throughout a topic or concept is not chartered. There is no mention of strategies, or thinking that learners use in order to achieve the required outcomes in the curriculum. This leaves teachers free to figure out the best way to achieve these outcomes (Daro, Mosher & Corcoran, 2011). The trade-off for autonomy in reaching these outcomes may be uncertainty on the part of the teachers, as they may not be able to, or do not have the time to devise the best paths to these outcomes.

Learning trajectories, especially in mathematics, is an area of research which focuses on learners' progression in their thinking about concepts. Wilson, Mojica and Confrey (2013) suggest that learning trajectories are very useful in helping teachers to understand how learners progress in their mathematical thinking, a process which may improve teaching practices.

Ronda (2004) has created an empirically based conceptual framework which describes learners' development of their understanding of functions, and refers to this path of development as learning trajectories. This *Framework of Growth Points* maps out 'big ideas' which learners typically encounter on their path to understanding functions (Ronda, 2009: p. 31). These growth points are in approximately the order that they are expected to encounter these ideas. Ronda (2009) suggests that most learners follow learning paths or trajectories, and reach growth points, as they develop their understanding of 'big ideas' of functions. My research uses Ronda's (2004) study into growth points and their description as its basis.

In testing Grade 9, 10 and 11 learners from a typical school in South Africa, I aimed to find out about learning trajectories in functions, and particularly in the area of functions when they are represented by an equation. I have used Ronda's (2004) framework of growth points to guide this research, and compare the way in which learners in South Africa progress through the growth points set out by Rhonda in her study which was done in Australia and the Philippines.

My research also deals with the language associated with the growth points, and how learners' language and use of language changes as they progress along their path of understanding. Caspi and Sfard's (2012) describe the progression through levels of discourse as canonic. This means that there is a hierarchy of levels of algebraic discourse, where each level of discourse builds on the previous level, and is hence more complex than the previous level (a meta-discourse of the previous level). Caspi and Sfard (2012) state that "transition from one level to another can be seen as developmental milestones", and hence this links with meeting Ronda's growth points on a learning trajectory.

In this paper, I will show that the learning of functions is not a quick process where an outcome can be achieved, but rather a process whereby learners reach growth points along which they progress systematically.

I also hope to show that certain types of language can be associated with the growth points. Teachers will be able to use this to better inform their teaching of functions to learners.

## REFERENCES

- Caspi, S. & Sfard, A. (2012). Spontaneous meta-arithmetic as a first step towards school algebra. *International Journal of Educational Research*. 51-52, 45-65
- Daro, P., F. Mosher, T. Corcoran. (2011). *Learning trajectories in mathematics: A foundation for standards, curriculum, assessment, and instruction*. Consortium for Policy Research in Education.
- Ronda, E. (2004). A Framework of Growth Points in students' developing understanding of function. Unpublished PHD: Australian Catholic University.
- Ronda, E. (2009). Growth Points in Students' Developing understanding of functions in equation form. *Mathematics Education Research Journal*, 21(1), 31-35.
- Wilson, P. H., Mojica, G. F., & Confrey, J. (2013). Learning trajectories in teacher education: Supporting teachers' understandings of students' mathematical thinking. *The Journal of Mathematical Behavior*, 32(2), 103-121.



# A REVIEW OF RESEARCH ON THE EFFECT OF USING A GROUP LEARNING APPROACH IN MATHEMATICS CLASSROOMS

Joseph J Dhlamini

*Department of Mathematics Education, University of South Africa*

*Educational research is in the forefront of investigating and evaluating the influence of a group learning approach on the scholastic performance of learners. In this paper a group learning approach refers to an arrangement in which two or more learners work together to achieve a common educational goal. The aim of this paper is to present a review of research comparing the effectiveness of using a group learning approach on the performance of learners in mathematics. The results of the review suggest that a group learning approach is mostly favoured for constructing powerful learning environments that impact positively to learning. The paper also discusses some aspects considered to account for the beneficial influence of group work in academic environments. A cognitive load theory is briefly explored to provide insights on how group learning activities have a potential to influence cognitive processes that are perceived to be at play during a problem-solving activity. In conclusion, the author emphasizes a need to place learners in powerful learning environments, such as those generated by group learning approach, to engage learners in effortful interactions that promote shared knowledge and understanding.*

**Key words:** Group learning approach; collaborative learning; cognitive load theory; mathematics performance

## INTRODUCTION

In almost every education system efforts are being made to find reformed ways to provide support to individual learners, and also find ways for effective collaboration. Recently, collaborative or group approach presents itself as the most explored channel of delivery to enhance and facilitate learning activities. In this paper the term *group learning approach* is used to refer to the learning environments in which learners work as a collective. In fact, a group learning approach is a general term that could be used to describe a teaching format in which learners are grouped heterogeneously within a school arrangement. In Dhlamini and Mogari (2013) the terms *collaborative group learning* and *a group approach* are used interchangeably to simply refer to “an arrangement in which two or more people work together to achieve a common goal” (p. 1). This, in turn, is expected to allow for rich problem analysis and quick arrival at the solution stage. Usually, within a group learning arrangement learners get an opportunity to work in groups of two or more members who mutually search for understanding, solutions, meaning and creative strategies to solve problems.

A stronger argument for employing a group learning approach is that of increasing learners' opportunities for effective interaction and meaningful sharing of ideas, thus increasing opportunities for an active mind in multiple contexts (Sanna Järvelä, Piia Näykki, Jari Laru & Tiina Luokkanen, 2007).

In a group learning setting the role of a teacher as a sole expert transmitter of knowledge is defused as learners engage in the exploration of the learning material on their own. However, the word '*difused*' as used in this context should not, by any means, be construed as implying the supportive role and the presence of the teacher in group learning environment disappear entirely. In the right way a good teacher can play a role of a facilitator or that of a motivator. For example, Slavin (1991) noted that it is not enough to simply tell learners to work together, and that learners should be made to have a reason to take one another's achievement seriously. In addition, Slavin (1991) developed a model that focuses on external motivators, which reside outside the group, such as rewards and individual accountability that are established by the teacher. His meta-analysis found that group tasks with structures promoting individual accountability produce stronger learning outcomes (Slavin, 1996). Therefore a teacher continues to remain a key component of any teaching experience, irrespective of the teaching approach that characterizes those interactions. In a group learning environment learners are guided by the teacher to take charge of their education. It is thus fitting to describe group learning environments as a significant shift away from the conventional teacher-dominated instruction (pedagogy) to that which is learner-centred. In the conventional teacher-dominated classrooms a teacher always tells learners information, which they are later expected to remember. However, in group learning environments learners construct their own knowledge (Cobb, Wood & Yackel, 1990).

In this vain a group learning approach could also be seen as a teaching arrangement that promotes active learning. In Carlson and Winqvist (2011) the phrase 'active learning' is used to refer to an act of asking learners to "do something" (p. 3). This explanation seems well in line with the goals of implementing a group learning approach, where learners could be given a mathematical problem to do and solve collectively. A group learning approach brings up an important feature to the role of classroom atmosphere and culture by opening up opportunities for group members to do something that contributes to their effective learning and specifically to the development of their intellectual autonomy as they are tasked to evaluate the worth of solution methods and provide justifications in collaborative discussion. In that way a group learning environment may provide learners with "effective tools to reinforce their problem-solving system" (Dhlamini, 2012, p. 241) that create a useful space for expanding their problem knowledge, social interaction and abilities to present and defend their views.



This is possible because the processes that occur during group discussions include verbalising explanations, justifications and reflections (Beers, Boshuizen & Kirschner, 2007; Kirschner, Beers, Boshuizen & Gijsselaers, 2008), giving mutual support (Van Boxtel, Van der Linden & Kanselaar, 2000) and developing arguments about complex problems (Munneke, Andriessen, Kanselaar & Kirschner, 2007). Dhlamini (2012) emphasises three elements of group learning activities: discussion, argumentation and reflection. As a characteristic feature, learners in a group setting get opportunities to argue out problem steps and solutions using their varied learning perspectives. The role of argumentation and the dynamics of social interactions in learning group settings have been widely discussed in literature (for examples, see, Dhlamini & Mogari, 2013; Hershkowitz & Schwarz, 1999; Whitenack & Yackel, 2002; Yackel & Cobb, 1996; Yackel, Wood & Cobb, 1993).

Given this background, this paper aims to provide a review of the literature of studies that focussed on the role of collaborative group work in the teaching and learning environments, as well as its influence on the learning outcomes. In most of the reviewed studies the pedagogical activities are grounded on the notion of group learning, which largely embraces the idea that “knowledge is constructed by learners based on their social (i.e., collaboration) and cognitive (i.e. problem-solving; self-regulation) activities” (Kester & Paas, 2005, p. 690). Based on this review, this paper goes further to provide a discussion covering some of the aspects of group work that are considered to account for its beneficial influence in learning environments. Also, a cognitive load theory (CLT) is used in this paper to provide insights on the workings of a human cognitive system, and how this system may be influenced when learning takes place within a group learning environment.

## **RESEARCH ON THE ROLE OF GROUP LEARNING APPROACH ON LEARNERS’ PERMANCE IN MATHEMATICS**

In almost every country around the world a need to design powerful teaching strategies and methods to elevate learners’ performance in mathematics has been identified. This need emanates from the realization and acknowledgement of challenges and demands of life and circular work in the twenty-first century. Multidisciplinary teams are used in industry, government and in education to solve complex problems to allow different perspectives to enrich the problem space. As such employees are constantly expected to: demonstrate effective communication skills; to collaborate; to negotiate and argue critically. How do we prepare our learners for these higher-order skills in order to expand meaningful labour participation in future? One way could be to encourage schools to construct learning environments that promote sustained learner engagement to develop effective collaborative skills (Dhlamini & Mogari, 2013). Given this background, there is a need to look into studies that were conducted to determine the academic influence of group work in teaching and learning environments.

Dhlamini (2012) conducted a quasi-experimental study in which 783 Grade 10 mathematics learners participated. All participants performed poorly in mathematics problem-solving. To demarcate the influence of different instructional approaches on learners' performance two teaching environments were constructed. In experimental schools (n=413) the researcher constructed learning environments that largely embraced aspects of group work by learners, while in control schools (n=370) teachers preserved conventional teaching conditions which were mainly teacher-dominated (Dhlamini & Mogari, 2013). Given the design of the study the mathematics performance of learners in both conditions was compared at pre- and post-stages on the experiment. The study found that the group approach, which was mainly implemented in experimental schools, appeared to be superior to the conventional teaching approaches implemented in the control schools in substantially improving learners' performance in certain topics in Grade 10 financial mathematics (see, Dhlamini & Mogari, 2013).

A study by Kirschner, Paas, and Kirschner (2009) explored the relationship between effort and performance when solving mathematics problems. The study involved high school learners who were learning from solving high-complexity mathematics tasks. The learners were organised in a group learning environment and an individual learning environment. The Kirschner, Paas, and Kirschner (2009) found a more favourable relationship between effort and performance in the learning phase for high school learners who worked in a group environment than for learners who worked individually. Invariably, there are several research studies which were conducted to examine group learning activities in elementary and secondary schools, which focussed on mathematics classrooms. The implication of these studies is that the use of group learning approach leads to better group productivity, improved attitudes, and increased performance in mathematics (Garfield, 1993).

In his study, Shaughnessy (1977) found that the use of group work approach appeared to help learners to overcome some misconceptions about probability, and in addition, group interactions enhanced the learning of statistical concepts. In another study Jones (1991) introduced group learning activities in several sections of statistics and later observed dramatic increases in attendance, class participation, office visits by learners, and improved learner attitude, which are all variables linked to performance. There are many other studies that have shown a positive impact on learning when learners participate in mathematics lessons that embrace elements of group and collaborative work (for examples, see, Dhlamini & Mogari, 2013; Newmann, 1996). For instance, Newmann (1996) found that learners in group learning environments learn better than those in individual learning environments. Barron (2000a, 2000b; 2003) conducted a series of quasi-experimental studies that compared the mathematics problem-solving performance of Grade 6 learners in group and individual environments.



Almost all studies found that learners in group learning environments significantly outperformed those in individual learning environments. In accounting for the observed performance difference, the studies noted that when learners were given a new analogous problem task to solve, those who had earlier solved the problem in group environments performed at a significantly higher level.

Complex Instruction (CI) is one of the best known and widely researched instructional approaches that are common in collaborative and group learning environments. As a teaching approach CI could be built on a set of carefully designed learning activities that require diverse talents and interdependence among group members (Cohen & Lotan, 1997). According to Cohen and Lotan (1997), within a CI approach teachers are encouraged to pay attention to unequal participation among group members, which often results from status differences among peers, and are given strategies that allow them to bolster the status of infrequent contributors. Within this teaching environment different roles, such as recorder, reporter, materials manager, resource manager, communication facilitator and harmonizer, are assigned to learners to promote and support equal participation. There is a strong body of research evidence supporting the success of CI generated instructions in promoting learners' academic performance in mathematics (for examples, see, Abram, Scarloss, Holthuis, Cohen, Lotan & Schultz, 2001; Cohen, 1994a, 1994b; Cohen & Lotan, 1995; Cohen et al., 1999, 2002).

One of the key elements in group learning environments is the role played by the teacher. In group learning environments a good teacher can play a role of a facilitator or that of a motivator. For example, Slavin (1991) noted that it is not enough to simply tell learners to work together, and that learners should be made to have a reason to take one another's achievement seriously. In addition, Slavin (1991) developed a model that focuses on external motivators that are established by the teacher, which reside outside the group learning environment, such as rewards and individual accountability. Consequently, the meta-analysis conducted in Slavin (1996) found that group tasks with structures promoting individual accountability produce stronger learning outcomes. Therefore a group learning approach becomes more efficient when a teacher allocate roles accordingly.

## ASPECTS OF A GROUP LEARNING APPROACH THAT ACCOUNT FOR ITS BENEFICIAL LEARNING EFFECT

Research has gone beyond just reporting merely the beneficial influence of group and collaborative approach in learning environments, it has however also reported on the aspects of this approach that are responsible for learning gains. For instance, a number of socially-oriented processes that play themselves out during group learning interactions have been identified as explaining why group work promotes individual learning. The social interactions that promote effective learning in group environments include: (1) the sharing of original insights and understanding (Bos, 1937); (2) resolving differing and opposing perspectives through argument (Amigues, 1988; Phelps & Damon, 1989); (3) explaining one's thinking about a particular phenomenon (King, 1990; Webb, Troper & Fall, 1995); (4) providing a useful and helpful critique (Bos, 1937); (5) observing other learners' strategies; (6) listen in order to provide an explanation later (Coleman, 1998; Hatano & Iganaki, 1991; Webb, 1985; Webb, 1985; Shirouzu, Miyake & Masukawa, 2002).

A series of experimental studies by Barron (2000a, 200b; 2003), which were reported earlier in this paper, reported more learning gains in group learning environments as opposed to learning environments that are exclusively individual-oriented. Further analysis in this studies also showed that the quality of the collaboration, that is, how learners talked and interacted with one another in group learning environments, was directly related their group score and later to their individual scores. Richards (2003) acknowledged that a group member's knowledge and expertise is not enough for the group to obtain the desired results; individual approaches should also be considered. Richards (2003) believed that effective groups require individuals who use and value different approaches when solving problems. According to McClough and Rogelberg (2003), group members that approach problems in a similar manner have relatively small amounts of tension, but may not produce the best solution. Therefore the type of approach that a group chooses to employ has a potential to determine the success rate in terms of how the problem solution is achieved.

In their summary of studies which were conducted over 40years on cooperative or group learning, Johnson and Johnson (1999) identified five basic elements of successful group learning. These are: positive interdependence, individual accountability, structures that promote face-to-face interaction, social skills, and group processing (Johnson and Johnson, 1999). In the same vain, Graves and Graves (1990) cite the following as the basic indicators of successful group learning: face-to-face heterogeneous learning teams, positive interdependence, individual accountability, explicit training in interpersonal skills and reflection. Social interactions that are normally encouraged in group learning environments provide opportunities for expression and debate by learners.

## THEORETICAL FOUNDATIONS OF THE PAPER

In Dhlamini and Mogari (2013) a cognitive load theory is used to inform the design of efficient group-based learning environments. Cognitive load refers to the mental burden and effort that an individual endures whilst executing a problem-solving task (Chen, 2003), and is largely linked to a *working memory*, which is considered to be critical in determining the success of learning. Working memory or the ‘short-term memory’ is the part of the memory, or human cognitive architecture that is needed to process incoming information (Kirschner, 2002). However, the limitations of the working memory in processing ability and duration are well documented and widely accepted within cognitive science research (Dhlamini & Mogari, 2011). Concerning its processing duration, researchers such as Paas, Van Gog and Sweller (2010) argue that almost all information stored in working memory and not rehearsed is lost within 30 seconds. Also, the working memory’s capacity cannot deal with more than about seven elements of information simultaneously (Engle, 2010). Hence, if the working memory capacity is exceeded whilst processing information then some or all information may be lost. When the working memory is unable to deal with or process information, the cognitive load may be said to be too high.

Although cognitive load theory generated lessons are meant to manage individual working memory load (cognitive load) of individuals (Kirschner, Paas, Kirschner & Janssen, 2011), Kirschner, Paas and Kirschner (2009) have emphasised an alternative technique of effectively dealing with individual working memory limitations by making use of the multiple working memories of individuals in group approach learning environments. From a cognitive load theory perspective, it is argued that dividing the processing of information in the working memory across individuals in a group approach environment is useful because this technique allows information to be divided across a larger reservoir of cognitive capacity, thus increasing the working memory capacity (Kirschner et al., 2011). According to Dhlamini (2012), in a group approach learning environment, which represents a huge working memory system, the limitations of individual working memories are not exposed because individual working memories are not subjected to processing each piece of problem-solving information. Therefore the risk of overloading each group member is lowered, and an individual’s working memory capacity is freed up whilst the group’s collective working memory capacity is expanded, and the cognitive load may be reduced (Kirschner, 2009).

According to Kirschner (2009), within a group setting information processing is characterised by active and conscious sharing (i.e. retrieving and explicating information), discussing (i.e. encoding and elaborating the information) and remembering (i.e. personalising and storing the information) valuable task-relevant information and knowledge held by each group member. For a group to perform a mathematics task, it is not necessary that all group members be highly knowledgeable in task-related information or be able to process all available information by themselves and at the same time (Johnson, Johnson & Stanne, 2001). As long as there is communication and coordination between group members, the information elements within the task and the associated cognitive load can be shared amongst group members (Kirschner, 2009).

## DISCUSSION AND CONCLUSION

Using research from other studies this paper has highlighted the important role of group work in teaching and learning environments, and the need to consider its inclusion and application in mathematics classrooms. It is true that recent curriculum recommendations include the use of group learning activities as a form of active learning to supplement or even replace conventional instructional approaches. Unfortunately, collaborative and group learning environments are not as heavily advocated in schools as one might think, and in some instances are discouraged in favour of conventional formats. Given the discussions in this paper there is a need to encourage mathematics teachers to embrace the philosophy of teaching some of the topics in mathematics using a group learning approach. In order to enhance learners' performance, this paper demonstrated that the cognitive load theory (CLT) could provide useful guidelines for instructional designs that are suitable to group learning environments.

The review in this paper focussed on studies that documented the positive influence of group learning approach on the performance of learners. While there has been this promising outcomes on the role of group learning environments on learners' performance, other researchers have contested that group learning is not necessarily a guarantee for positive learning outcomes (see, Boylan & Smith, 2012; Gregor & Cuskelly, 1994; Heath, 1998; Soller, 2001). For instance, Boylan and Smith (2012) reported that collaborative or group learning environments could create many challenges for learners. Using CLT principles, Kirschner, Paas, and Kirschner (2009b) have argued that the possible cause of studies showing mixed and even negative findings may be that the structures constituting cognitive architecture have not been systematically considered when designing and carrying out research on group learning (for other inputs, see also Dhlamini & Mogari, 2011; Kirschner, Sweller, & Clark, 2006).



In addition, this paper has reiterated the important role of a teacher in group learning environments, and that this critical role should not be discarded. Given some of the discussions in this paper it is clear that the teacher could play a critical role in establishing and modelling productive group norms and practices to promote meaningful group conversations. While helping learners to organize themselves in groups of five to seven members a teacher may also help learners to set out rules to regulate group dynamics, and further assist group members to assign roles to each group members. Learners engaged in such group learning environments cannot only strengthen their content knowledge but can also learn increasingly important twenty-first century skills, such as the ability to work in teams, solve complex problems and to apply knowledge gained in group setting to a variety of circumstances. Therefore cooperative group work is an ideal environment for learners to construct knowledge through social interactions. Cobb, Wood and Yackel (1990) note that social interactions constitute a crucial source of opportunities for learners to learn mathematics through constructing individual's mathematical knowledge.

The author believes that a group learning approach should be made a popular learning arrangement for teaching mathematics in schools. In mathematics classrooms, group learning approach seems to hold a great promise as a supplement to other modes of subject delivery by providing learners with opportunities to practice mathematics skills and concepts with peers, and using mathematical language to discuss and critique mathematical concepts. In conclusion, it is the author's view that mathematics learners should be placed in powerful learning environments, such as those generated by group learning approach, to engage them in effortful interactions that promote shared knowledge and meaningful understanding of mathematics.

#### REFERENCES

- Abram, P., Scarloss, B., Holthuis N., Cohen, E., Lotan R., & Schultz, S. E. (2001). The use of evaluation criteria to improve academic discussion in cooperative groups. *Asian Journal of Education*, 22, 16-27.
- Amigues, R. (1988). Peer interaction in solving physics problems: Socio-cognitive confrontation and metacognitive aspects. *Journal of Experimental Child Psychology*, 45(1), 141-158.
- Barron, B. (2000a). Achieving coordination in collaborative problem-solving groups. *Journal of the Learning Sciences*, 9(4), 403-436.
- Barron, B. (2000b). Problem-solving in video-based microworlds: Collaborative and individual outcomes of high-achieving sixth-grade students. *Journal of Educational Psychology*, 92(2), 391-398.
- Barron, B. (2003). When smart groups fail. *Journal of the Learning Sciences*, 12(3), 307-359.
- Beers, P. J., Boshuizen, H. P. A., & Kirschner, P. A. (2007). The analysis of negotiation of common ground in CSCL. *Learning and Instruction*, 17, 427-435. <http://dx.doi.org/10.1016/j.learninstruc.2007.04.002>.
- Bos, M. C. (1937). Experimental study of productive collaboration. *Acta Psychologica*, 3, 315-426.
- Boylan, M., & Smith, P. (2012) Tutors in collaborative group work. *Student Engagement and Experience Journal*, 1(1), 1-13.
- Boylan, M., & Smith, P. (2012). Tutors in collaborative group work. *Student Engagement and Experience Journal*, 1(1), 1-13.

- Carlson, K. A., & Winquist, J. R. (2011). Evaluating an active learning approach to teaching introductory statistics: A classroom workbook approach. *Journal of Statistics Education*, 19(1), 1-23.
- Chen, M. (2003). Cognitive load theory and its implications for instruction. *Educational Review of National Kaohsiung Normal University*, 21, 29-51.
- Cobb, P., Wood, T., & Yackel, E. (1990). Classroom as learning environments for teachers and researchers. In R. Davis et al. (Eds), *Constructivist views on the teaching and learning of mathematics*, A JRME Monograph Number 4: Reston, Virginia: NCTM.
- Cohen, E. G. (1994a). *Designing group work: Strategies for heterogeneous classrooms*, Revised edition. New York: Teachers College Press.
- Cohen, E. G. (1994b). Restructuring the classroom: Conditions for productive small groups. *Review of Educational Research*, 64(1), 1-35.
- Cohen, E. G., & Lotan, R. A. (1995). Producing equal-status interaction in the heterogeneous classroom. *American Educational Research Journal*, 32(1), 99-120.
- Cohen, E. G., & Lotan, R. A. (Eds.). (1997). *Working for equity in heterogeneous classrooms: Sociological theory in practice*. New York: Teachers College Press.
- Cohen, E. G., Lotan, R. A., Abram, P. L., Scarloss, B. A., & Schultz, S. E. (2002). Can groups learn? *Teachers College Record*, 104(6), 1045-1068.
- Cohen, E. G., Lotan, R. A., Scarloss, B. A., & Arellano, A. R. (1999). Complex instruction: Equity in cooperative learning classrooms. *Theory into Practice*, 38(2), 80-86.
- Coleman, E. B. (1998). Using explanatory knowledge during collaborative problem-solving in science. *Journal of the Learning Sciences*, 7(3-4), 387-427.
- Dhlamini, J. J. (2012). *Investigating the effect of context-based problem-solving instruction on the performance of learners*. Unpublished doctoral dissertation. University of South Africa, Pretoria, South Africa. Available from <http://hdl.handle.net/10500/9739>.
- Dhlamini, J. J., & Mogari, D. (2011). Using a cognitive load theory to account for the enhancement of high school learners' mathematical problem-solving skills. In D. Mogari, A. Mji, & U. I. Ogbonnaya (Eds.), *Proceedings of the ISTE 2011 International Conference: Towards Effective Teaching and Meaningful Learning in MST* (pp. 309-326). Kruger National Park, Mpumalanga: University of South Africa.
- Dhlamini, J. J., & Mogari, D. (2013). The effect of a group approach on the performance of high school mathematics learners. *Pythagoras*, 34(2), Art. #198, 9 pages. <http://dx.doi.org/10.4102/pythagoras.v34i2.198>.
- Engle, R. W. (2010). Role of working memory capacity in cognitive control. *Current Anthropology*, 51(1), S17-S26. <http://dx.doi.org/10.1086/650572> Erichsen.
- Garfield, J. (1993). Teaching statistics using small-group cooperative learning. *Journal of Statistics Education*, 1(1), 1-12.
- Gregor, S. D., & Cuskelly, E. F. (1994). Computer-mediated communication in distance education. *Journal of Computer Assisted Learning*, 10, 168-181.
- Hatano & Ignaki. (1991). Sharing cognition through collective comprehension activity. In L. B. Resnick, J. Levine, & S. Teasley (Eds.), *Perspectives on socially shared cognition* (pp. 331-348). Washington, DC: American Psychological Association.
- Heath, E. F. (1998). Two cheers and a pint of worry: An on-line course in political and social philosophy. *Journal of Asynchronous Learning Networks*, 2, 15-33.
- Järvelä, S., Näykki, P., Laru, J., & Luokkanen., T. (2007). Structuring and regulating collaborative learning in higher education with wireless networks and mobile tools. *Educational Technology and Society*, 10 (4), 71-79.



- Johnson, D. W., Johnson, R. T., & Stanne M. B. (2001). Impact of goal and resource interdependence on problem-solving success. *The Journal of Social Psychology, 129*, 621-629. <http://dx.doi.org/10.1080/00224545.1989.9713780>.
- Jones, L. (1991). *Using cooperative learning to teach statistics*. Research report number 91-92, The L.L. Thurstone Psychometric Laboratory, University of North Carolina.
- Kester, L., & Paas, F. (2005). Instructional interventions to enhance collaboration in powerful learning environments. *Computers in Human Behavior, 21*(1), 689-696.
- King, A. (1990). Enhancing peer interaction and learning in the classroom through reciprocal peer questioning. *American Educational Research Journal, 27*(4), 664-687.
- Kirschner, F. C. (2009). *United brains for complex learning: A cognitive load approach to collaborative learning efficiency*. Unpublished doctoral dissertation. Inter-university Centre for Educational Research, Open University, Maastricht, The Netherlands. Available from <http://dspace.ou.nl/handle/1820/2391>.
- Kirschner, F. C., Paas, F., & Kirschner, P. A. (2009). A cognitive load approach to collaborative learning: United brains for complex tasks. *Educational Psychology Review, 21*, 31-42. <http://dx.doi.org/10.1007/s10648-008-9095-2>.
- Kirschner, F. C., Paas, F., Kirschner, P. A., & Janssen, J. (2011). Differential effects of problem-solving demands on individual and collaborative learning outcomes. *Learning and Instruction, 21*, 587-599. <http://dx.doi.org/10.1016/j.learninstruc.2011.01.001>.
- Kirschner, F., Paas, F., & Kirschner, P. A. (2009). Individual and group-based learning from complex cognitive tasks: Effects on retention and transfer efficiency. *Computers in Human Behavior, 25*, 306-314.
- Kirschner, F., Paas, F., & Kirschner, P. A. (2009b). A cognitive load approach to collaborative learning: United brains for complex tasks. *Educational Psychology Review, 21*, 31-42.
- Kirschner, P. A. (2002). Cognitive load theory: Implications of cognitive load theory on the design of learning. *Learning and Instruction, 12*, 1-10. [http://dx.doi.org/10.1016/S0959-4752\(01\)00014-7](http://dx.doi.org/10.1016/S0959-4752(01)00014-7).
- Kirschner, P. A., Beers, P. J., Boshuizen, H. P. A., & Gijselaers, W. H. (2008). Coercing shared knowledge in collaborative learning environments. *Computers in Human Behaviour, 24*, 403-420. <http://dx.doi.org/10.1016/j.chb.2007.01.028>.
- Kirschner, P. A., Sweller, J., & Clark, R. E. (2006). Why minimal guidance during instruction does not work: An analysis of the failure of constructivist, discovery, problem-based, experiential, and inquiry-based teaching. *Educational Psychologist, 46*, 75-86.
- Munneke, L., Andriessen, J., Kanselaar, G., & Kirschner, P. A. (2007). Supporting interactive argumentation: Influence of representational tools on discussing a wicked problem. *Computers in Human Behaviour, 23*(3), 1072-1088. <http://dx.doi.org/10.1016/j.chb.2006.10.003>.
- Newmann, F. M. (1996). *Authentic achievement: Restructuring schools for intellectual quality*. San Francisco: Jossey-Bass.
- Paas, F. G. W. C., Van Gog, T., & Sweller, J. (2010). Cognitive load theory: New conceptualizations, specifications, and integrated research perspectives. *Educational Psychology Review, 22*(1), 115-121. <http://dx.doi.org/10.1007/s10648-010-9133-8>.
- Phelps, E., & Damon, W. (1989). Problem-solving with equals: Peer collaboration as a context for learning mathematics and spatial concepts. *Journal of Educational Psychology, 81*(4), 639-646.
- Shaughnessy, J. M. (1977). Misconceptions of probability: An experiment with a small-group activity-based model building approach to introductory probability at the college level. *Educational Studies in Mathematics, 8*, 285-316.
- Shirouzu, H., Miyake, N., & Masukawa, H. (2002). Cognitively active externalization for situated reflection. *Cognitive Science, 26*(4), 469-501.

- Slavin, R. E. (1991). Synthesis of research on cooperative learning. *Educational Leadership*, 71-82.
- Slavin, R. E. (1996). Research on cooperative learning and achievement: What we know, what we need to know. *Contemporary Educational Psychology*, 21(1), 43-69.
- Soller, A. L. (2001). Supporting social interaction in an intelligent collaborative learning system. *International Journal of Artificial Intelligence in Education*, 12, 40-62.
- Van Boxtel, C. A. M., Van der Linden, J., & Kanselaar, G. (2000). Collaborative learning tasks and the elaboration of conceptual knowledge. *Learning and Instruction*, 10, 311-330. [http://dx.doi.org/10.1016/S0959-4752\(00\)00002-5](http://dx.doi.org/10.1016/S0959-4752(00)00002-5).
- Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.
- Webb, N. M. (1985). Verbal interaction and learning in peer-directed groups. *Theory into Practice*, 24(1), 32-39.
- Webb, N. M., Troper, J. D., & Fall, R. (1995). Constructive activity and learning in collaborative small groups. *Journal of educational psychology*, 87(3), 406-423.



## EXPLORING TEACHERS' NOTION OF LOGICAL THINKING

Roland Fray

Department of Mathematics, University of the Western Cape

*The aim in this paper is to explain the outcome of teachers' hands-on experience of solving logical puzzles and their opinions on the usefulness of this in the teaching of geometry.*

### INTRODUCTION

The aim in this article is to share my experiences in teaching a short course in Euclidean Geometry to educators in the Northern Cape province of South Africa. The Science Learning Centre for Africa at the University of the Western Cape (UWC-SLCA) was contracted by the Northern Cape Education Department and funded by the ETDP- SETA to offer the course in Kimberley recently.

At the start of the course teachers were asked individually to share their experiences in teaching geometry with the rest of the group which consisted of educators of mathematics in the Further Education and Training band (FET band) of the National Senior Certificate, that is, grades ten to twelve. It was especially interesting to hear about the different methods that teachers employed to teach the subject. One teacher likened the solving of a geometry rider to the work of a detective trying to solve a crime that was committed. Another teacher thought of the process of solving a geometry problem as unwinding a piece of wool that became entangled by first finding one endpoint and then to continue the unwinding by untying the knots in the string until you arrive at one long string with the endpoint representing the solution.

One teacher used the method of identifying key words in a rider that related to a theorem or axiom. What is popular from what other teachers described is the method of colouring in on the sketch with colour pens.

Some of the comments made by teachers attending the short course about the learners included:

1. Learners can't think logically.
2. Learners switch off when geometry is taught.
3. Learners have a negative attitude towards geometry.
4. Learners are lazy to think.
5. Learners' inability to read geometry problems with understanding was also mentioned. In this regard the question was raised by one teacher whether learners know and understand the terminology used in the statement of the problem. Another teacher remarked that this could also be ascribed to learners' general reading ability.

These comments are perhaps inter-related with the most important one, in my opinion, being learners' inability to think logically. Among the many reasons for this might be that geometry in the FET phase requires fairly sophisticated thinking processes which are lacking in many learners as a result of not being exposed to this much earlier. For example, it is expected of learners to:

1. Reproduce proofs of theorems and their converses which require the ability to write down their arguments in full mathematical sentences with each statement accompanied by a reason.
2. Integrate their knowledge in solving problems, that is, learners must be able to apply many results in a single problem. It goes without saying therefore that learners must know all the definitions of concepts, theorems as well as their converses (referred to as bookwork).
3. Be familiar with the processes of proving and to understand why proofs are required in mathematics. [Proving a theorem in geometry is very much like putting the pieces of a puzzle together. As in the case of a puzzle where the pieces are placed in position one at a time, the details of a proof must be written down step by step. Each step should be justified by a definition, axiom (mathematical principle), a known result (a theorem or lemma) or some algebraic manipulations.]
4. Understand how new theorems can be discovered and to construct proofs independently and thereby have an appreciation of the ever expanding nature of knowledge.

From the above it should be obvious that it is a complex issue to teach learners how to prove mathematical statements in geometry, in particular. This is because of the many facets that one needs to deal with such as, for example, the ability of learners to read what is required in the problem, the conceptual understanding of the learners, the ability of the learners to carry out algebraic manipulations correctly, the ability of the learners to write their answers in a logical sequence that is coherent and clear as well as the ability of the learners to make appropriate constructions when needed.

It was not clear from teachers' explanations what they meant when they say that learners can't think logically. My impression was that most teachers attending the course had an intuitive idea of what they meant. The aim in this paper is to explain the outcome of teachers' hands-on experience of solving logical puzzles and their opinions on the usefulness of this in the teaching of geometry. It is hoped that educators can use this as a tool in their teaching and thereby introducing learners to the thinking processes required in problem-solving in general, but in Euclidean Geometry, in particular.



## Logical Puzzles

At the start of the second session teachers were asked to solve the following puzzles taken from the book by Epp,S, Discrete Mathematics with Applications, Second Edition [3].

- (a) The famous detective Percule Hoirot was called in to solve a baffling murder mystery. He determined the following facts:
1. Lord Hazelton, the murdered man, was killed by a blow on the head with a brass candlestick.
  2. Either Lady Hazelton or a maid, Sara, was in the dining room at the time of the murder.
  3. If the cook was in the kitchen at the time of the murder, then the butler killed Lord Hazelton with a fatal dose of strychnine.
  4. If lady Hazelton was in the dining room at the time of the murder, then the chauffeur killed Lord Hazelton.
  5. If the cook was not in the kitchen at the time of the murder, then Sara was not in the dining room when the murder was committed.
  6. If Sara was in the dining room at the time the murder was committed, then the wine steward killed Lord Hazelton.

Is it possible for the detective to deduce the identity of the murderer from the above facts? If so, who did murder Lord Hazelton? (Assume there was only one cause of death.)

- (b) In the back of an old cupboard you discover a note signed by a pirate famous for his bizarre sense of humour and love of logical puzzles. In the note he wrote that he had hidden treasure somewhere on the property. He listed five true statements (1 – 5 below) and challenged the reader to use them to figure out the location of the treasure.
1. If this house is next to a lake, then the treasure is not in the kitchen.
  2. If the tree in the front yard is an elm, then the treasure is in the kitchen.
  3. This house is next to a lake.
  4. The tree in the front yard is an elm or the treasure is buried under the flagpole.
  5. If the tree in the backyard is an oak, then the treasure is in the garage.
- Where is the treasure hidden?
- (c) You are about to leave for school in the morning and discover you don't have your glasses. You know the following statements are true:

1. If my glasses are on the kitchen table, then I saw them at breakfast.
2. I was reading the newspaper in the living room or I was reading the newspaper in the kitchen.
3. If I was reading the newspaper in the living room, then my glasses are on the coffee table.
4. I did not see my glasses at breakfast.
5. If I was reading my book in bed, then my glasses are on the bedside table.
6. If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table.

Where are the glasses?

### Results

Most teachers attending the course who had the correct answer to a puzzle did not provide any logical reasoning which explained how they got to the answer. There were however a few good attempts where there was clear evidence of logical thinking in the arguments. We will analyze some of these in section 5. We summarize the results obtained by educators in the table below.

	PUZZLE 1	PUZZLE 2	PUZZLE 3
Correct Answer	12	21	14
Incorrect Answer	31	22	28

Selected teachers' responses to the question:

### What do you think is the relevance, if at all, of solving puzzles to the teaching of geometry?

Educator 1:

- Dit leer die kinders om alle gegewe inligting te lees en te interpreter.
- Om korrekte afleidings te maak.
- Die feite van die valshede te onderskei(contradictions).
- Om gegewe inligting in verband met mekaar te plaas.
- Om onwaarhede te kansleer.
- Om ekstra inligting wat geen verband hou met die problem te identifiseer en te elimineer.
- Identifiseer die problem
- Om volgende te bepaal.



- Do not jump to conclusion without proof.
- An interesting way of introducing the thinking skills(logical skills) used in geometry.

Educator 2:

- Identify facts.
- Eliminate unwanted information.
- Reasoning abilities are stimulated.
- Interpretation of statements.
- Read with insight.
- Motivate their answers.
- Logical order justify their answers.

Educator 3:

- Leerders word gestimuleer om logies te redeneer, oplossings te vind en geometry solutions vereis problem-oplossing.
- Logical thinking, reasoning and to find solutions to problems.
- They learn to cancel out some information/ rule out certain facts to come to right answer.
- Kyk na belangrike inligting → probleemoplossing.
- It is very important to read the information given with understanding and then to identify the facts/ keywords that are relevant in order to solve the problem.
- Eliminate unnecessary information- avoid confusion.

Educator 4:

- Assisted with reading with interpretation.
- Help to form conclusions from contradictory statements.
- Help to eliminate irrelevant information.
- Assists to deduce a true statement
- Help to formulate a conclusion by means of elimination.
- Moves from the unknown to the known.
- Filtering process (like a funnel) to get rid of the unnecessary information.

**Analysis of teachers’ attempts at solving the three puzzles.**

From the results in section 3, it is evident that the majority of the teachers could not solve the puzzles. This demonstrated that educators in the course would benefit a lot from learning more about elementary logic with special emphasis on valid and invalid argument forms. We analyse the attempts of four educators at solving the three puzzles. In the following table we give the solutions for each of the four educators accompanied by my comments at each step of the solution indicating the logical rules of inference that are applied. This indicates that, even though there are some gaps in their arguments some educators attending the course applied logical thinking principles in their solutions of the three puzzles.

Table of four educators solutions of the puzzles:

Educator 1:

ANSWER TO PUZZLE 1	COMMENT
From 5, cook not in kitchen.	Give a reason: only one cause of death ∴ The Butler did not kill Lord Hazelton with a fatal dose of strychnine. ∴ Cook not in kitchen [# 3,modus tollens].
∴ Sara not in dining room	#5, modus ponens.
∴ Lazy Hazelton in dining room	From 2, disjunctive syllogism
∴ Chauffeur killed Lord Hazelton	From 2 and 4, modus ponens.
	#6 represents redundant information.

ANSWER TO PUZZLE 2	COMMENT
From 3 House next to lake	
∴ treasure not in kitchen	# 1, modus ponens
Since treasure not in kitchen elm tree statement not valid	Tree in front yard is not an elm #2, modus tollens
From 4 treasure is buried under the flagpole	#4, disjunctive syllogism
	#5 represents redundant information



ANSWER TO PUZZLE 3	COMMENT
From 4 Did not see glasses at bfast.	
∴ glasses not on kitchen table	From #1 and #4, modus tollens
	∴ I was not reading the newspaper in the kitchen [from #6, modus tollens ∴ I was reading the newspaper in the living room [from 2, disjunctive syllogism
∴ Read newspaper in living room can't be in bedroom because was used to read newspaper	Where does this come from?
∴ left it on coffee table	#3, modus ponens
	#5 represents redundant information.

Educator 2:

ANSWER TO PUZZLE 1	COMMENT
If the cook was in the kitchen the murder weapon does not match the one found	Only one cause of death. ∴ The Butler did not kill Lord Hazelton with a fatal dose of strychnine
So the cook was not in the kitchen	From 3, modus tollens
Which means Sara was not in the dining room	From 5, modus ponens
Which means the Lady was there	From 2, disjunctive syllogism
And that means the chauffeur killed the Lord	# 4, modus ponens
	#6 represents redundant information

ANSWER TO PUZZLE 2	COMMENT
The house is next to a lake	From 3
Therefore the treasure is not in the kitchen	#1, modus ponens
Which means that there is not an elm tree in front of the yard.	From 2, modus tollens
Therefore the treasure is buried under the flagpole	From 4, disjunctive syllogism
	#5 represents redundant information.

ANSWER TO PUZZLE 3	COMMENT
I did not see my glasses at breakfast	# 4
Therefore the glasses are not on the kitchen table	# 1, modus tollens
Therefore I was not reading in the kitchen	#6, modus tollens
But in the living room	#2, disjunctive syllogism
Which means that my glasses are on the coffee table	#3, modus ponens
I could not been reading in the bedroom cause I either read in the living room or the kitchen therefore the glasses cannot be on the bedside.	#5 represents redundant information



Educator 3:

ANSWER TO PUZZLE 1	COMMENT
There was only one cause of death so the butler didn't kill Lord Hazelton with a dose of strychnine. If the butler didn't do it the cook was not in the kitchen	# 3,modus tollens
And Sara not in the dining room	#5,modus ponens
If Sara was not in the dining room Lady Hazelton was in the dining room	#2,Disjunctive Syllogism
The chauffeur killed Lord Hazelton	#4, modus ponens
	#6 represents redundant information
ANSWER TO PUZZLE 2	COMMENT
The house is next to the lake	#3
So the treasure is not in the kitchen	#3 and #1 modus ponens
So if treasure is not in kitchen the tree in front of yard is not an elm	#2, modus tollens
So if the tree in front of yard is not an elm the treasure is buried under the flagpole	#4, disjunctive syllogism
	#5 represents redundant information.
ANSWER TO PUZZLE 3	COMMENT
I did not see my glasses at breakfast	#4
So my glasses is not on kitchen table	#1 and #4, modus tollens
So if my glasses is not on kitchen table I did not read the newspaper in the kitchen	#6, modus tollens
If I did not read the newspaper in the kitchen I was reading the Newspaper in the living room	#2,disjunctive syllogism
So if I was reading in the living room my glasses is on the coffee table	#3, modus ponens
	#5 represents redundant information.

Educator 4:

ANSWER TO PUZZLE 1	COMMENT
The chauffeur killed him.	
1) Either Lady Hazelton or Sara was in the dining room at the time of the murder. 2) So if the cook was in the kitchen then he killed him with a fatal dose of strychnine and that was not the cause of death	$\therefore$ The butler did not kill Lord Hazelton with a fatal dose of strychnine $\therefore$ The cook was not in the kitchen
So if the cook was not in the kitchen	
Sara was not in the dining room	#5 ,modus ponens
Which leaves Lady Hazelton	#2, disjunctive syllogism
And if she was in the room the chauffeur killed him	#4, modus ponens.
	#6 represents redundant information.

ANSWER TO PUZZLE 2	COMMENT
The treasure is buried under the flagpole	
Statement #3 states that the house is next to the lake	
So it is not in the kitchen	#1,modus ponens
#4 tree=elm= in kitchen which we ruled out with #3	Not #4 but #2
	$\therefore$ from 2 using modus tollens tree not an elm
So in order to have 5 true statements we go with the “or” which is under the flagpole.	$\therefore$ #4, disjunctive syllogism
	#5 represents redundant information.



ANSWER TO PUZZLE 3	COMMENT
In the living room on the coffee table.	
It's not in the kitchen because #1 states that if it's there then you saw it	
#4 states that you did not see it	$\therefore$ glasses not on kitchen table, #1 and # 4, modus tollens
	$\therefore$ I was not reading the newspaper in the kitchen #6, modus tollens
Number 2 states that you were reading in the living room or kitchen	
So that only leaves the living room on the coffee table	#3, modus ponens

The analysis of the solutions as was done above, show clearly the logical rules that the four educators apply, perhaps unknowingly or they might have this ability which distinguishes them from the rest of the group. It is possible for all the participants to learn how to solve such puzzles once they've learnt some elementary logic.

### CONCLUSION

Very few teachers could answer all three puzzles correctly. Many wrote the correct answer but were unable to provide a logical explanation. Only five out of the 43 teachers who worked on the puzzles showed evidence of logical thinking in their explanations of their answers. From the analysis of four educators' solutions, it is clear from my comments that the educators applied logical rules of inference in their solutions. Some of their arguments had gaps and it is evident that educators were aware of redundant information that only serves the purpose of confusing the person trying to solve the puzzle. Some of the teachers requested more puzzles. In fact one of the educators reported that her daughter said that learners would enjoy solving such puzzles. The educators' remarks on the usefulness of such puzzles in the teaching of geometry were all positive. It would be interesting to learn whether the teachers are using this in their teaching.

It is unfortunate that due to time constraints we could not give the participants a lot more practice on solving such puzzles

## REFERENCES

Bittinger, Marvin L.(1982). Logic, Proof, and Sets 2<sup>nd</sup> Edition (Addison-Wesley Publishing Company)

Caroll,L. Symbolic Logic and the Game of Logic. New York: Dover Publications Inc.

Epp,S.S.(1995). Discrete Mathematics with Applications,2<sup>nd</sup> Edition. Belmont,California: Thompson Learning.

Solow, Daniel. (2002). How to read and Do Proofs (An introduction to mathematical thought processes) 3<sup>rd</sup> Edition John Wiley& Sons.



# FACTORS CONTRIBUTING TO THE POPULARITY OF MATHEMATICS OLYMPIADS AND COMPETITIONS IN SOME SCHOOLS: AN INTERROGATION OF LEARNERS' AND TEACHERS' VIEWS.

VG Govender

Eastern Cape Department of Education

*Problem-solving is one of the features that makes mathematics an important subject in South Africa and other countries. Although the new NCS R-12 emphasises the need for problem-solving in the classroom, it would appear that teachers are not doing enough in this part of the mathematics curriculum. The writer has found that some affluent schools tend to provide problem-solving opportunities for their top learners by encouraging them to participate in Mathematics Olympiads and competitions. The data emerging from this study suggests that there are certain factors which contribute to the popularity of Mathematics Olympiads and competitions in some schools. These factors include the role of the teacher, learning culture of the school, performance in school mathematics, participation in team events and the role of the parent.*

## INTRODUCTION

Mathematics is a key subject in South Africa and other countries. In the National Curriculum Statement (NCS R - 12) document, commonly known as the CAPS document, mathematics is described as follows:

Mathematics is a language that makes use of symbols and notations for describing numerical, geometric and graphical relationships. It is a human activity that involves observing, representing and investigating patterns and qualitative relationships in physical and social phenomena and between mathematical objects themselves. It helps to develop mental processes that enhance logical and critical thinking, accuracy and problem-solving that will contribute in decision- making. Mathematics problem-solving enables us to understand the world (physical, social and economic) around us, and most of all, to think creatively. (DBE, 2011)

This description of Mathematics is very inclusive and highlights important mathematical content areas such as numbers, geometry, graphs and patterns. It also highlights important mathematical skills such as observing, representing and investigating. Embedded in these content areas and skills is one of the cornerstones of the mathematics curriculum, that of problem-solving.

It is expected, as envisaged in the description above, that problem-solving should be a key feature of mathematics teaching and learning. However, there tend to be challenges with regard to the promotion of real problem-solving in South African mathematics classrooms. There are general complaints by teachers that the South African mathematics curriculum is too full and there is very little or no time to pursue real problem-solving during mathematics lessons. The more affluent schools tend to overcome this challenge by enrolling their learners as participants in the various mathematics problem-solving competitions or Olympiads. I have been involved in initiatives to promote Mathematics Olympiads among the less affluent schools. Unfortunately, due to a myriad of factors, these initiatives have not had the desired effect.

There are various Mathematics Olympiads or competitions which target school learners. One of these is organised under the banner of the Govan Mbeki Mathematics Development Unit (GMMDU) located at the Nelson Mandela Metropolitan University (NMMU). It is stated in its website of the GMMDU that this competition mainly targets learners in the Eastern Cape and aims to “**improve the problem-solving skills of learners**”. It also aims to **promote awareness of the importance of Mathematics and Applied Mathematics** in society. It emphasises the importance of the **problem-solving approach to mathematics as “central to the modern way of teaching mathematics”** GMMDU, 2014).

The South African Mathematics Olympiad (SAMO) is organised under the banner of the South African Mathematics Foundation (SAMF). It targets both junior high school learners (grades 8 & 9) and senior high school learners (grades 10 -12) (SAMO, 2014). In South Africa and other counties, Mathematics Olympiads are used to identify talented learners of Mathematics. In some instances, the focus is on discovering learners with specific aptitudes such as verbal or quantitative and developing their talent. (Assouline & Lupkowski-Shoplik, 2012)

## STATEMENT OF THE PROBLEM

I have had many years of experience as an examiner or co-examiner of mathematics problem-solving competitions or Olympiads. I have found, from personal experience, that learners who are successful in these competitions are usually from the more affluent or advantaged schools. As stated earlier, previous initiatives to promote Mathematics Olympiads among disadvantaged schools have not been as successful as one would like it to be.



When I had to work with top Mathematics Olympiad learners in preparation for the South African Interprovincial Mathematics Olympiad (SAIPMO) in 2013, I used this opportunity to find out more from these learners and, thereafter their teachers about their involvement in Mathematics Olympiads. These learners were selected from a list of top learners, provided by the South African Mathematics Foundation (SAMF), in round 2 of the South African Mathematics Olympiad (SAMO). I felt that by interacting with these learners and teachers, I would probably find out more from them about their participation in Mathematics Olympiads. This could be used to promote Mathematics Olympiads at other schools.

### RESEARCH QUESTION

In the light of the aforementioned discussions, I posed the following research question for this study:

#### **What factors contribute to the popularity of Mathematics Olympiads in some schools?**

To answer the research question the following subsidiary questions were formulated in the context of the research question:

- Is there a relationship between participation in Mathematics Olympiads and performance in school mathematics?
- What are some of the factors which impact positively on learner participation in Mathematics Olympiads?
- What role do teachers play in promoting Mathematics Olympiads amongst their learners?
- How do Mathematics Olympiad learners work in group questions in an interprovincial competition and what strategies are most successful?

### SAMPLE

There were 22 learners and 5 teachers in this sample. The learner group consisted of both junior learners (grade 8 & 9) and senior learners (grades 10 – 12). The teachers were selected from the same schools as the learners. Both learners and teachers participated willingly in this research.

### CONCEPTUAL FRAMEWORK

In South Africa, Mathematics Olympiads are an “add-on” for learners and teachers. Learners tend to work through Olympiad- type questions in their own time and their teachers may or may not give them support. However, these Olympiads tend to enrich the learners’ mathematical knowledge. Schools may use Mathematics Olympiads and competitions to promote excellence in mathematics learning, develop and enhance self confidence in learners and also nurture creativity amongst the learners.

In this regard, the Enrichment Triad Model (ETM) may be an appropriate framework for this study (Renzulli, 1977). This model consists of three different kinds of interrelated forms of enrichment activities that are integrated as a complement to the regular curriculum.

Type I enrichment consists of general exploratory experiences that are designed to expose students to topics and areas of study not ordinarily covered in the regular curriculum. Type II enrichment consists of group training in thinking and feeling processes, learning-how-to-learn skills, research and reference skills, and written, oral, and visual communication skills. Type III enrichment consists of first-hand projects or investigations intended to solve real problems (Renzulli, 1999). For the purposes of this research, type I and type III enrichment may be relevant while type II would probably surface during the team paper of the SAIPMO.

## **RESEARCH METHODOLOGY**

Hatch (2002) reports that for qualitative researchers the experiences of real people in real settings are the objects of study. This type of inquiry is concerned with understanding how individuals make sense of their everyday lives. Qualitative research also seeks to understand the world from the perspectives of those living in it. This was very relevant to this research where the focus was on Mathematics Olympiads and an attempt was made to understand Mathematics Olympiads from both the learners' and teachers' perspectives.

Both the learners and their teachers completed questionnaires which covered the issues outlined in the research questions and sub-questions. As this study was conducted during the South African Interprovincial Mathematics Olympiad (SAIPMO), the quantitative data in this study was the learners' individual and group marks in the SAIPMO.

## **THE LEARNER QUESTIONNAIRE**

In order to get a deep understanding of learners' perspectives on their participation in Olympiads, the following information was sought in their questionnaires:

Performance in school mathematics and possible reasons for such performance; experience in Mathematics Olympiads; reasons for participating in Mathematics Olympiads; mathematics role models; giving advice to others; and other comments: In order to cover issues which may not have come up in the previous categories, I left it open for learners to give other comments.

## **THE TEACHER QUESTIONNAIRE**

The teachers in the sample were from the same schools as the learners selected for the SAIPMO. They had to provide the following information in their questionnaires:

Involvement in Mathematics Olympiads over the years; the value of Mathematics Olympiads; the school's policy in respect of Mathematics Olympiads; training sessions for learners; parental involvement and support; advice to other schools on Olympiads and their participation in different Mathematics Olympiads/competitions.



## **THE SOUTH AFRICAN INTERPROVINCIAL MATHEMATICS OLYMPIAD (SAIPMO)**

Teams from all over South Africa and Southern Africa participate in SAIPMO. I was called by a SAMF representative in July 2013 to enter two teams (one junior and one senior) in the SAIPMO of 2013. I was given a list of top learners in round two of the South African Mathematics Olympiad. I made my selection of the teams from the lists provided. The SAIPMO provided this research with two sets of important data; the first one being the preparation for the SAIPMO and the second was the performance of the two teams in SAIPMO.

### **THE DATA**

The data from the questionnaires were analysed with a view to detecting trends and patterns of coherence. Where appropriate, some interesting individual comment(s) are also highlighted.

### **THE LEARNER QUESTIONNAIRE**

#### **Performance in mathematics**

Both the junior and senior team members had done very well in school mathematics over the years. These learners were at the top in their mathematics classes and their marks in mathematics tended to be in excess of 95%. These learners were always working hard and striving for mathematics excellence. Most of them received awards for mathematics. They loved the subject and were always encouraged and supported by parents. They also had dedicated teachers.

### Experience in taking part in Mathematics Olympiads

The next two tables show the earliest grades which these learners started participating in Mathematics Olympiad or Competitions.

Grade	Number of learners
3	4
4	2
5	0
7	3
9	1
12	1 (immigrant)
Total	11

Table 1: Senior team first grades of participation

Grade	Number of learners
3	5
4	3
5	0
7	1
8	2
Total	11

Table 2: Junior team first grades of participation

Of the 22 learners in the sample, 14 of them participated in Mathematics Olympiads or competitions for the first time in grade 3 or grade 4. Thus, the majority of these learners had a very early exposure to Olympiads or competitions.



## Reasons for participating in Mathematics Olympiads

Both the senior team and junior teams gave the following reasons for liking Mathematics Olympiads:

- They enjoyed working with high level, challenging and interesting problems.
- It gave them an opportunity for achieving excellence and making it into further rounds of the Olympiads.
- By achieving well in the Olympiads, there was an opportunity of winning prizes.
- These learners liked mathematics and found Mathematics Olympiads enjoyable and entertaining. For them it was a “good” challenge.
- Olympiads helped develop their problem-solving skills and expand their mathematical knowledge.
- Their participation in Olympiads enhanced their CVs (senior team only).

## Mathematics role models

Four of the senior team members listed their mathematics teachers as role models. These role models inspired them to do well in mathematics. An interesting choice of one learner was “himself” stating that he did not have any role models and that he always challenged himself to do well in Olympiads. Only one senior learner chose a parent as a role model.

One learner claimed to have been inspired by “past students who have excelled in Olympiads”. This tied in with the choice of another learner who was very specific about his role models. He chose Bruce Merry and Liam Baker. I did internet searches on both these men. Bruce Merry is a Computer Science specialist from the University of Cape Town and has received gold, silver and bronze medals in the International Olympiad on Informatics in the years 1996 -2001 (Merry, 2014). Liam Baker became the first South African to have a problem proposal selected for inclusion in the International Mathematics Olympiad (IMO), when his functional equation was one of six used at the 2012 edition of the competition for the sharpest high-school mathematics brains in the world (University of Cape Town Centre for Higher Education Development, 2014).

One could see why these choices were made. This learner, who was in grade 12 at the time, participated in International Olympiads in Mathematics and Informatics and received South African colours. Clearly, the feats of Bruce Merry and Liam Baker left an indelible mark on this learner. An interesting choice by another learner with surname Newton was Isaac Newton. This learner said: His work inspires me to continue the family tradition.

Seven of the junior team members cited one or both parents as being their mathematics role models who inspired them to participate and do well in Mathematics Olympiads. The majority of these (five) chose the mother. Other choices were teacher (chosen by two), uncle and brother. It is evident here that the mothers played a very important role in these younger (grade 8 or 9) learner's lives. As is evident from the senior team's responses, this may change as they got older.

### **Giving advice to other children**

It would appear from the responses of both the senior and junior teams that they were keen for other children to have similar experiences when it came to participation and success in Mathematics Olympiads. This is very encouraging and could provide a platform for schools to introduce Mathematics Olympiads or competitions to their learners. In this regard members of the senior team gave the following advice:

- Children should prepare well and not be put off or give up.
- They should test their ability and add to what they learn in class.
- They should challenge themselves and enjoy the opportunity.
- Children should learn to think “outside the box”.
- They should think carefully and look for different ways of solving problems.

The junior team suggested the following:

- Children should try their best and never give up.
- They should give themselves much needed practice by working through a number of different types of examples.
- They should use different problem-solving methods.
- Children should not spend too much time on one problem.
- They should always be striving for a higher level in mathematics.

### **Other comments**

Both the senior and junior teams commented that Mathematics was an important school subject but school mathematics did not give learners adequate opportunity to develop their problem-solving skills. Thus, learners should be given the opportunity of participating in Mathematics Olympiads. This would enable learners to use important skills such creativity and insight, when working through Olympiad type problems. However, as this was an on-going process, they cautioned that learners participating in Olympiads for the first time should be realistic about their expectations.



## THE TEACHER QUESTIONNAIRE

For purposes of data analysis, the five schools in question were called A, B, C, D and E.

### The number of years the school has been involved in Mathematics Olympiads

Schools which encourage learners to participate in Mathematics Olympiads usually have a long and proud tradition of participation. The inclusion of this question in the teacher questionnaire was to check whether this was true for these schools.

School	Number of years
A	34
B	15
C	25
D	1
E	10

Table 3: Years of participation in Mathematics Olympiads

Four of the five schools had a long participation in Olympiads, ranging from 10 to 34 years, confirming my assertion in the previous paragraph. Only school D had participated for one year.

### The value of Mathematics Olympiads

All teachers in the sample were very aware of the value of Mathematics Olympiads for their learners. In this regard, they provided the following reasons as to why participation in Mathematics Olympiads was important:

- It provided learners an opportunity to improve their problem-solving skills by encouraging deep, lateral, independent and creative thinking.
- It allows learners to enjoy and engage with mathematics in a different context. Gifted and talented learners are exposed to challenging problems and given the opportunity to excel.
- During practice sessions, usually without teachers, learners are able to discuss different problem-solving strategies and to “think outside the box.”
- It helps learners become confident and develop inquiring minds.

### **The school's policy in respect of Mathematics Olympiads**

None of the schools in question had any specific policy with respect to participation in Mathematics Olympiads. At each school, there is a teacher in charge and everything is left to the teacher. Sometimes, student teachers are roped in to help. At some schools, the Olympiad dates are put on the school year plan and important dates are passed onto learners. Although all mathematics learners are encouraged to participate in the Olympiads, usually only the top learners do so.

### **Training sessions for learners**

The teachers reported that no specific training sessions are arranged for their learners in respect of Mathematics Olympiads. Their learners work on their own as teachers are very busy with full teaching loads. Teachers may download past papers which are then passed on to the learners. Sometimes, learners may approach their teachers for support.

### **Parental involvement and support**

Teachers stated that parents were very positive about their children taking part in Mathematics Olympiads and competitions. They are always very willing to provide transport. They welcomed opportunities for their children to participate in competitions in a social setting where group-work or working in pairs is encouraged. Some parents also tend to encourage their children to be competitive and aim to win prizes that may be on offer. The teachers also reported that parents do not expect them to provide training or tutorial sessions.

### **Advice to other schools on Mathematics Olympiads**

To promote problem-solving amongst learners, teachers should incorporate problem-solving activities in their teaching. This is a gradual but important process. After learners are sufficiently accustomed to problem-solving, teachers should enter a few learners in low key events. In this regard, the inter-school mathematics relay, organised by school. Teachers in the sample were full of praise for this inter-school mathematics relay, stating that the relay was fun and competitive.

According to one teacher:

Most maths teachers at government schools do not feel very confident about helping pupils with Olympiads at senior level and do not have the time to spend preparing sufficiently to help their pupils.

The participation of learners in team events would be less daunting for them. This would increase motivation and emphasise the need for collaborations amongst learners.



### **Participation in various Mathematics Olympiads and Competitions**

All schools participated in the South African Mathematics Olympiad (SAMO) which has a junior section (grades 8 & 9) and senior section (grades 10 – 12). SAMO takes place over three rounds. In addition to this Olympiad, schools A, B and C participated in the NMMU/AMESA Mxit Mathematics Competition for junior learners (grade 8 & 9) and senior learners (grade 10 & 11). This competition also took place over three rounds.

Schools A and E also took part in the University of Cape Town (UCT) Mathematics Olympiad (grades 8 -12) and the University of Pretoria Mathematics Olympiad (grade 8-11). Another competition was the Young Mathematicians convention, organised by the International Montessori School for grades (10 – 12). School A participated in this competition.

As stated earlier, school A had a long tradition of participation in Mathematics Olympiads and other competitions. This school had a very dedicated mathematics teacher who also organised the inter-school mathematics relay.

### **PARTICIPATION IN THE SAIPMO**

The first part of SAIPMO, which lasted an hour, consisted of Olympiad questions which learners had to answer as individuals. There were 15 questions in this part of SAIPMO. Thereafter, learners had a 30 minute break. During this time, learners had some refreshments and then both senior and junior team members started discussing strategy for the team paper.

The team paper consisted of 10 questions, each question carrying 100 marks. In this part of SAIPMO teams had to give only one set of responses. There were very intense and deliberate discussions among both groups of learners. To capture the marks scored by learners in the individual part of SAIPMO, the members of the teams were named as J1 (junior team member 1); J2 (junior team member 2); S1 (senior team member 1) and so on.

Tables 4 and 5 show the individual marks for the SAIPMO.

Team member	Marks obtained
J1	66
J2	76
J3	76
J4	68
J5	78
J6	51
J7	42
J8	56
J9	44
J10	39
<b>TOTAL</b>	<b>596</b>

Table 4: Junior team individual marks

Team member	Marks obtained
S1	100
S2	62
S3	31
S4	78
S5	52
S6	55
S7	62
S8	40
S9	85
S10	27
<b>TOTAL</b>	<b>592</b>

Table 5: Senior team individual marks

There were some notable performances in the individual rounds, especially for the senior team. In the team paper, the junior team scored 400 points while the senior team scored 700 points. The overall points for the junior team were:  $596 + 400 = 996$ . The junior team was placed 19<sup>th</sup> out of 34 teams nationally. The overall points for the senior team were:  $592 + 700 = 1292$ . The senior team was more impressive in that it was placed 3<sup>rd</sup> out of 34 teams. This is shown in table 6.



Team	Individual	Team	Total	Rank out of 34
Junior	596	400	996	19
Senior	592	700	1292	3

Table 6: Team scores and rank in SAIPMO

## DISCUSSION

The learners and teachers participated in this research very enthusiastically and provided rich data. The learners were selected from very good schools and usually performed very well in their school mathematics, obtaining marks above 90%. These learners were looking for mathematics that challenged them; hence, their participation in Mathematics Olympiads and competitions.

They liked taking part in Mathematics Olympiads as this tended to build on their problem-solving skills. Most of the learners were frequent participants in Mathematics Olympiads. The junior learners tended to regard a parent (especially a mother) as an inspiring role model while the senior learners' choices were more diverse. All learners had clearly enjoyed participating in Mathematics Olympiads and wanted learners from other schools to benefit from this experience. They felt that these learners would also develop their problem-solving skills and creativity.

The teachers were very aware of the importance of Mathematics Olympiads and its role in the mathematical development of learners. Most of the schools had more than 10 years of participation in Mathematics Olympiads and competitions. However, there was no specific school policy on Mathematics Olympiads. Teachers usually played a coordinating role and provided learners with the dates and other such information. Due to the heavy work-loads of the teachers, none of them were able to have formal training sessions for their learners. One teacher, however, stated that learners were able to ask her for assistance when needed.

The schools participated in a number of Mathematics Olympiads and Competitions. This included the inter-school relay competition which was organised by school A. These teachers believed that other schools should also encourage their learners to participate in Mathematics Olympiads and competitions. A group competition such as the Inter-school relay would be an ideal vehicle to ease learners into such participation.

At the various schools, Mathematics Olympiads are regarded as an “add-on” and a way of enriching the mathematics of the learners involved. Thus, schools encourage their learners to participate in a variety of Mathematics Olympiads and competitions. Many of these have prizes or certificates as incentives and serve as a motivation for learners at these schools.

From an interrogation of the data and the discussion in this section, I would now like to outline my findings from this research. This is done in conjunction with my research sub-questions and research question posed earlier in this paper.

## **THE RESEARCH SUB-QUESTIONS**

### **Is there a relationship between participation in Mathematics Olympiads and performance in school mathematics?**

The learners in this research had done very well in round 1 and round 2 of the SAMO. These learners were selected to participate in the SAIPMO, based on their results in round 2 of the SAMO. These learners were the top performers both in the city and in the province. These learners revealed to me in their questionnaires that they performed very well in their school mathematics, getting marks in the region of 95% plus. Thus, it would appear from this sample of learners that, there is a strong relationship between participation in Mathematics Olympiad and performance in school mathematics.

### **What are some of the factors which impact positively on learner participation in Mathematics Olympiads?**

The learners in this research were top Olympiad learners who have mostly been involved in Olympiads and competitions for a number of years. They were located in schools which had a long participation in these Olympiads and competitions. They had teachers who provided them with information and regular updates on Mathematics Olympiads and competitions. They had very supportive parents who probably encouraged them and transported them to Olympiad venues, when necessary. In this regard these factors may be summarised as follows:

Top performance in school mathematics; length of exposure to Mathematics Olympiads and competitions; previous success in Mathematics Olympiads and competitions and supportive teachers and parents

### **What role do teachers play in promoting Mathematics Olympiads amongst their learners?**

It would appear that the teachers in this small sample were very modest about their role in promoting Mathematics Olympiads among their learners. In fact, I believe that they tended to downplay their roles. They pointed out that they would share dates with learners and ensure that Olympiads were included in the school year programme or plan. I venture to say that if these teachers did not do this, then their learners would not be participating in Olympiads. The inclusion of Olympiads in the school year programme ensured that Olympiads were regarded as an integral part of the school programme (as an enrichment activity).



I would also say that despite having full teaching loads, these teachers would be regarded as great “ambassadors” for Mathematics Olympiads at their schools. They ensure that their learners in their schools are given exposure to many Olympiads and this has probably resulted in an “Olympiad” culture at their schools. In many instances, these teachers play a lone hand in promoting Mathematics Olympiads at their schools.

### **How do Mathematics Olympiad learners work in group questions in an interprovincial competition?**

The junior and senior teams had similar performances in the individual paper but in the team paper, the teams used different strategies. The senior team worked in pairs. Each pair worked through two questions and then submitted their answers to the team captain. The team captain had the final say on the submission of the answers. The junior team worked as individuals and then submitted their answers to the team captain. This strategy appeared to be more cumbersome and the selection of the final answers did not go very well.

It would seem that learners who participate in Mathematics Olympiads are very good at working on their own. However, when working in a team, team strategy and tactics must be discussed and spelt out clearly. In this research, while both junior and senior teams had similar performances in the individual questions, it would appear that the senior team strategy worked better during the team questions. The senior team had 7 out of 10 questions correct.

One area where team strategy could be developed is during Mathematics relays. The teachers in the sample were avid supporters of the inter-school relay for Mathematics, organised by school A. They claimed this relay, organised at both junior and senior levels, would be an ideal vehicle to expose schools to Mathematics competitions. In addition, learners work with mathematics in a social setting and have an opportunity of developing team strategy and tactics.

### **What factors contribute to the popularity of Mathematics Olympiads and competitions in some schools?**

After analysing my findings in terms of my research sub-questions, it is now possible to answer the research question. In this regard the data collected in this study, and an interrogation thereof, points to the following factors which contribute to the popularity of Mathematics Olympiads and competitions in some schools.

#### **The role of the teacher cannot be underestimated:**

In this research, the data pointed to a single teacher being the driving force behind Mathematics Olympiads at each of the schools in question. These teachers usually work alone and are instrumental in informing learners about dates and placing the Olympiad in their school’s year programme. These duties are carried out despite having full teaching loads.

### **The learning culture of the school:**

It is evident from this research that conditions should exist within a school to make participation in a Mathematics Olympiad or Competition mandatory for top learners and others who are interested. Four of the five schools had 10 or more years of participation in Mathematics Olympiads. Thus, Mathematics Olympiads have become an integral part of the learning culture of these schools. In other words, it has become a “normal” activity.

### **The performance of learners in their school mathematics:**

In this research, the learners were usually at the top of their classes. However, these learners were looking for challenges and this was provided by the Mathematics Olympiads. After having participated in Olympiads for many years, these learners saw the value of such participation and its influence on their own mathematics development. On this issue, I would say that it is imperative for all schools to encourage their top learners to participate in Olympiads and competitions. This will contribute to the all-round development of their top mathematics learners. This could also serve as a motivation for other learners to improve their school mathematics performance so as to be considered as a prospective participant in Mathematics Olympiads. However, the first rounds of many Mathematics Olympiads or competitions tend to focus on mass participation. In this case, schools should enter both their top learners and other learners who are interested. It is possible that this may unearth learners who may not be performing well in their mathematics but are good at problem-solving and thinking “out of the box”.

### **Participation in team events:**

This research showed that although there were similar individual performances for both the junior and senior team in the SAIPMO there were considerable differences in their team strategy. This resulted in different team performances. Teachers were full of praise for the Mathematics Inter-school relay competition held by school A. They believed that this would be less daunting and less intimidating for first time participants. At the same time, participation and success in the relays has the potential to make learners more confident about Mathematics competitions, in general. This could lead to increased participation in Mathematics Olympiads and competitions for individuals.

### **The involvement of parents:**

Teachers reported that parents are very supportive of their efforts to promote Mathematics Olympiads at their schools. I have also noticed this first hand. During the training sessions for the SAIPMO and the SAIPMO, itself, the majority of the learners were transported to the venue by their parents. Some parents also called me to inquire about the results of their children and the overall performance of the teams. I have also attended Olympiad prize-giving events where parents tend to support their children in large numbers.



## CONCLUSION

This research involving top Mathematics Olympiad learners and their teachers provided me with some rich data. Although the sample was small, certain factors were identified as contributing to the popularity of Mathematics Olympiads at these schools. These factors could be used by various stakeholders such as the Education Department, AMESA, SAMF and other organisations involved in Mathematics Olympiads and competitions to try to make these Olympiads and competitions accessible to more learners.

Popularising participation in Mathematics Olympiads and competitions is unlikely to be easy or quick. As seen in this research, it takes years of participation, a conducive learning environment, willing learners and dedicated teachers to ensure that curriculum enrichment programmes such as Mathematics Olympiads and competitions are implemented at more schools in South Africa.

## REFERENCES

- Assouline, S.G. & Lupkowski-Shoplik, initials. (2012). The talent search model of gifted identification. *Journal of Psychoeducational Assessment*, 30(1),45-49.
- Department of Basic Education (DBE) (2011): *National Curriculum Statement (NCS). Curriculum and Assessment Policy Statement (CAPS). Further Education and Training Phase. Grades 10 – 12. Mathematics*. Government Printing Works. Pretoria.
- Govan Mbeki Mathematics Development Unit (2011): *Mathematics Olympiad*. Nelson Mandela Metropolitan University (Faculty of Science). Retrieved January 13, 2014 from <http://mbeki-maths-dev.nmmu.ac.za/Projects/Mathematics-Olympiad>.
- Hatch, J. A., 2002. *Doing qualitative research in education settings*. Albany: State University of New York Press.
- Merry, B (home page) (2014). Retrieved February 21, 2014 from [www.brucemerry.org.za](http://www.brucemerry.org.za)
- Renzulli, J. (1977). *The enrichment triad model: a guide for developing defensible programs for the gifted and talented*. Wethersfield, CT: Creative Learning Press
- Renzulli, J. (1999). What is this thing called giftedness, and how do we develop it? A twenty-five year perspective. *Journal for the Education of the Gifted*, 23(1), 3-54.
- South African Mathematics Olympiad (SAMO) (2014). Retrieved January 12, 2014 from [www.samf.ac.za/Default2.aspx](http://www.samf.ac.za/Default2.aspx).
- University of Cape Town Centre for Higher Education Development (2014). Retrieved February 21, from [www.ched.uct.ac.za/news/archives/?id=9265&t=mp](http://www.ched.uct.ac.za/news/archives/?id=9265&t=mp).

# USING THE SOUTH AFRICAN MATHEMATICS CHALLENGE TO DEVELOP PRE-SERVICE MATHEMATICS TEACHERS' PROBLEM-SOLVING ABILITIES

VG Govender

Eastern Cape Department of Education

*Problem -solving is an integral part of the teaching and learning of mathematics. It is expected that mathematics teachers should be good problem solvers in their own right. However, this is not always the case with teachers, due to a lack of confidence and experience, among other factors. This research, conducted with pre-service mathematics teachers and using the South African Mathematics Challenge as a vehicle, has shown that it is possible to develop problem-solving skills and abilities in pre-service teachers. It may also provide a way in which these skills and abilities could be developed in in-service teachers.*

## INTRODUCTION

Teacher training in South Africa has undergone a number of reforms since the advent of democracy in South Africa. One of the most significant of these reforms was the decision by the then Minister of Education, Professor Kader Asmal to make teacher training in South Africa a competence of higher education institutions. Thus, from 1998 till 2001, there was a massive restructuring of teacher training, with all Colleges of Education, eventually being closed down.

One of the current challenges in teacher training in South Africa is the recruitment of competent students for such training and the quality of the programmes being offered, especially in subjects such as mathematics and physical sciences. This issue came to the fore in a report by the Ministerial Task Team on Mathematics, Science and Technology. One of its recommendations, listed as priority 2, was to “Address teachers and teaching issues”. The fourth bullet under this priority states the following about pre-service teacher education:

There is a critical need to intervene in pre-service teacher production in order to ensure that HEIs produce competent and credible new teachers of sufficient quality and in sufficient quantities to service the MST needs of the school system. (DBE, 2013:53)

The writer of this paper was involved as a coordinator of a B.ED (Science Option) programme at an Eastern Cape university, from 2011 till 2013. A close scrutiny of the Mathematics Method and Science Method modules revealed a lack of alignment with existing school practices. One of the first tasks was to reorganise the topics or content covered in these modules. At this university, the second and third year method students are exposed to senior phase (grade 7-9) mathematics content; FET content is covered in the fourth year.

Table 1 shows the topics that are included in the second year programme, after this reorganisation:

First Semester	Second Semester
1. Students' school experiences of Mathematics	1. Using pair and group work in mathematics
2. The nature of Mathematics	2. Problem- solving activities
3. Teaching methodologies	3. Overview of the senior phase mathematics content
4. Problem- based learning	4. Assessment
5. The number system	5. Geometry constructions
6. Estimation and number patterns	6. Geometry – investigations
7. Introduction to algebra	7. Ratio and proportion
8. Selected mathematics content: Simple equations; factorisation; word problems	8. Data handling
9. Lesson plans and microteaching	

Table 1: B.ED Mathematics Method Syllabus

A study of Table 1 shows that students are introduced to problem-based learning in the first semester and this is continued in the second semester under the heading “problem-solving activities”.

A survey of the second year students revealed that only one of 14 students had taken part in Mathematics Olympiads or competitions while at school. This was, indeed, an alarming and frightening statistic. It meant that these students' lack of exposure to the problems in Olympiads and Mathematics competitions were likely to impact on their own learners in the future. Thus, the inclusion of these topics in their method modules (in the first and second semesters) were designed to change this scenario and make sure these pre-service mathematics teachers were familiar with Olympiad-type problems.

The South African Mathematics Challenge is a competition for grades 4-7 learners of Mathematics. It is an AMESA (Association for Mathematics Education in South Africa) initiated competition which is organised by the South African Mathematics Foundation. The challenge is conducted in various regions of South Africa, with regional coordinators (usually AMESA members) responsible for liaising with schools and collecting statistics (South African Mathematics Challenge, 2014). In 2012, the writer of this paper was the regional organiser of the mathematics challenge in a rural town of the Eastern Cape. In the first round of the challenge approximately 900 learners from five primary schools participated.

The writer felt it opportune to use the grade 7 paper (first and final round) in this research as it was in line with some of the mathematics school content which had to be covered within the second year mathematics method module.

### RESEARCH QUESTION

Since this research involved ascertaining and improving the problem-solving ability of pre-service teachers, the following research question was posed for this study:

#### **Are pre-service teachers of Mathematics competent in solving Olympiad-type problems?**

In an attempt to answer the research question, the following subsidiary questions were formulated in the context of the study:

- How do pre-service students perform in mathematics competitions which target learners?
- Can the performance of pre-service students in mathematics competitions be improved through a structured intervention?
- What are the views of pre-service students on the mathematics competitions such as the Mathematics Challenge?
- Can pre-service students be encouraged to incorporate Olympiad type problems in their classrooms?

### SAMPLE

The sample for this study consisted of 14 students, all of whom consented to their participation in this study. Thus, the sample would be classified as “convenience sampling” (Skowronekand & Duerr, 2009).

### RESEARCH METHODOLOGY

Both quantitative and qualitative data were collected in this study. The quantitative data consisted of the students’ marks in both the first and final round of the South African Mathematics Challenge.

This study was about the experiences of pre-service teachers of mathematics with respect to Mathematics Olympiads and Competitions and to understand how they viewed these from their perspectives, making this study more qualitative in nature (Hatch, 2002).

The qualitative data consisted of the following:

- Students’ feedback on the South African Mathematics challenge (this was completed by 10 of the 14 students)
- A classification of the items in the South African Mathematics challenge according to certain categories (completed in groups)
- Comments about the type of questions in the challenge (completed in groups)
- How they would incorporate such questions in their normal classroom teaching (completed in groups)



- How they would prepare their learners for competitions such as the South African Mathematics challenge (completed in groups)

### OPERATIONAL STRATEGY

This study involved students writing both the first and final round of the grade 7 South African Mathematics challenge. In the period between the two rounds of the challenge, the students were involved in a number of activities. The students wrote the first round of the grade 7 paper. After completing the paper, they had to provide general feedback on the challenge. At the same time learners from the five primary schools wrote the grades 4 – 7 challenge. These papers were collected and brought to the university.

As part of a follow-up class activity the students marked the papers. From this activity, students became familiar with the types of questions in the challenge and the performance of the learners in the challenge. In the next Mathematics method lesson, students were divided into three groups. Each group had to conduct an in-depth analysis of the grade 7 paper followed by comments on the questions and how they would incorporate problem-solving type questions in their teaching. They also had to indicate how they would prepare their learners for mathematics competitions (see previous section on qualitative data).

### CONCEPTUAL FRAMEWORK

This research had both an ontological and epistemological basis (Cohen & Manion, 1985). The ontological basis was primarily about the experience of the students in writing South African Mathematical Olympiads and Competitions. It was known to the writer (via a survey) that only one student in the sample had previously written a Mathematics Olympiad while at school. Thus, it was clear to the researcher that the students in the Mathematics method class would need to be exposed to Mathematics Olympiads. The process of students acquiring this exposure would be in line with the epistemological basis of this research, which was to make the students aware of the nature of Mathematics Olympiads and Competitions. This would be done by using the South African Mathematics Challenge for grades 4 -7. This awareness would be acquired by students marking the papers of around 900 learners from the five primary schools and then making a detailed analysis of the grade 7 papers.

A suitable framework for this study would be in line with experiential learning theory. Experiential learning involves a number of steps that offer students a hands-on, collaborative and reflective learning experience which helps them to “fully learn new skills and knowledge” (Haynes, 2007). Although learning content is important, learning from the *process* is at the heart of experiential learning. During each step of the experience, students will engage with the content, the instructor, each other as well as self-reflect and apply what they have learned in another situation (Haynes, 2007). This was definitely in keeping with the students’ engagement with the various activities in this study.



## RESULTS

### Performance in first round

Table 2 below shows the performance of the students in the first round of the South African Mathematics Challenge for grade 7.

Student	First Round
A	9
B	24
C	4
D	7
E	6
F	6
G	7
H	10
I	13
J	7
K	19
L	16
M	18
N	14
<b>Average</b>	<b>11,43</b>

Table 2: Performance (out of 25) in the first round of the Mathematics Challenge  
Learners who participate in the Mathematics Challenge require 60% (15 out of 25) in the first round to qualify for the final round. By applying these criteria to the students in this sample, one finds that only four students would have qualified for the final round.



At the outset, these marks reflect poorly on the students. These students had passed grade 12 Mathematics (with a minimum of 50%), and also completed all the modules for first- year university mathematics. Yet, they were unable to perform well in a grade 7 paper. Clearly, these students had very little experience in working through problem-solving type questions and this lack of experience probably contributed to their poor performance.

### **STUDENTS' FEEDBACK ON THE FIRST ROUND**

The views of the 10 students who provided feedback on the paper are recorded here. The words written here are the words of the students. Some editing has been done to correct spelling and grammatical mistakes. The students are named A, B, C, etc. to ensure confidentiality.

Student B:

The test required a proper understanding of the basic concepts of mathematics. It made you think of better strategies that one should use when teaching these concepts. Such a test would also be designed to enable school children's ability to think and assess their level of understanding of basic mathematics, prior to going to high school.

Student C:

The paper was very challenging. It was difficult to answer some questions and I resorted to guessing. I must work harder and put in more effort to answer a grade 7 paper (at my level). Every question required logical reasoning and the use of mathematical skills. I did finish the paper but did not have time to check.

Student E:

Some of the questions may be confusing to grade 7 learners. I believe it is above their level of understanding and may be a bit abstract or complicated to them. They have to apply their knowledge and skills they have acquired from their experiences. They will perform well if they are exposed to these types of questions and can work on their own. These questions develop their level of understanding and problem-solving skills.

Student F:

It was a challenging paper. It is good for developing the student teachers' knowledge and skills in respect of problem -solving. They would be able to become better at problem -solving and can pass on these problem- solving skills to their learners.

Student H:

The first few questions were easy and this could have been to draw the learners' interest and to change their attitude. If learners were negative then they would think that the test was too difficult. The paper kept learners active and helped develop learners' calculating, analysing, observation and measuring skills.

Student I:

The paper is good for learners but they should get enough practice in these problems. The first few questions got the interest of the learners. Teachers must play an important role in teaching learners about problem -solving strategies and problem -solving.

Student J:

I found the word problems to be difficult. I tried my best but have developed a "negative" attitude towards word problems. I still have this problem up to this day. I thought that the paper was going to be easy but I was very surprised at the level of the paper. It involved critical thinking, knowledge of basic algebra and some abstract concepts. It was quite a high standard. The paper was good for learners' cognitive development and would make them better mathematics learners if they were exposed to these type of questions in their earlier grades.

Student K:

The test was fair and simple. It consisted of topics which learners were supposed to know or understand by grade 7. However, learners had to think critically and creatively and use various problem- solving skills when working through the questions in the paper. They had to deal with a number of questions involving patterns and had to use different strategies when working through these questions.

Student L:

The test was not what I expected. I was expecting less of a challenge but it was hard and challenging. It required logical reasoning and I thought the standard to be higher than that which was expected for primary school learners. The test did not require the use of a calculator. Learners needed to have a solid foundation in mathematics. I think the teaching of mathematics at the primary school is of a low standard and does not prepare learners for competitions such as the Mathematics Challenge. Even though the test appeared to be difficult, I believe it will "sprout" an awakening call to teachers to raise their standards in their mathematics classrooms.



Student M:

The paper was not easy. I struggled with geometric patterns. I believe that the paper would be difficult for grade 7 learners, especially if they have not been exposed to these questions. The challenge is good for them as they get exposed to different concepts. Teachers should incorporate these questions in their lessons and this would be beneficial to learners.

### TRENDS EMERGING FROM THE STUDENTS' VIEWS

There appeared to be some correlation between the students' marks in the paper and their comments. Those who did well had favourable comments while others found the paper to be difficult and challenging. Some of the key issues raised were:

- The paper was challenging for them and would be beyond the ability levels of primary school learners.
- The teacher is the key role player in the developing of thinking and problem- solving skills in learners. They should know more about critical thinking and problem- solving strategies and be able to expose their learners to these crucial mathematical skills.
- Problem- solving should be incorporated into mathematics lessons. If learners are exposed to problem-solving type questions in earlier grades, then they should have no trouble in working through these types of questions.
- The importance of a solid foundation in mathematics cannot be underestimated. In this regard, primary school teachers should raise their standard of teaching and making sure that their learners are assisted to achieve this standard.
- It is important for schools to encourage their learners to participate in mathematics competitions such as the Mathematics Challenge.

### MARKING OF THE PAPERS

The students were given an opportunity of marking approximately 900 first- round papers. They attempted this task very enthusiastically as they felt it would give them some practice in marking as part of their professional development. The marking was made easier for them as it was a multiple-choice paper and learners had to fill in their responses on a given template. A marking template of correct solutions for each of grades 4 -7 was developed. The marking of the papers proceeded smoothly with no issues. The only concern of the students was the poor performance of the learners. Only 18 learners had achieved the 60% cut-off to write the second round.

Such poor primary school performance in mathematics has been in the news lately with primary school learners not performing up to expectations in the grade 1 – 6 Annual National Assessments (ANA) for mathematics. The second part of the statement by student L, shown below, is relevant in this regard.

I think the teaching of mathematics at primary school is of a low standard and does not prepare learners for competitions such as the Mathematics Challenge. Even though the test appeared to be difficult, I believe it will “sprout” an awakening call to teachers to raise their standards in their mathematics classrooms.

It is possible that primary schools with a culture of participation in mathematics competitions have raised their standard of teaching mathematics and have probably had good ANA results in mathematics. This needs to be investigated further.

### **ANALYSIS OF THE GRADE 7 PAPER**

The students were divided into groups (on a random basis) and had to analyse each question and classify the question according to given categories. This activity was to enable students to become more familiar with the broad mathematics content that is covered in competitions such as the Mathematics Challenge. This would also serve as good practice in locating questions into the content areas of the Senior Phase curriculum (DBE, 2011). These broad categories and possible alignment to the Senior Phase mathematics curriculum (CAPS) is shown in table 3.



<b>Broad category for the mathematics challenge (grade 7)</b>	<b>Alignment to content areas of the Senior Phase mathematics curriculum</b>
---	--

- |   |                                       |
|---|---------------------------------------|
| 1. Numbers and number sense (whole numbers) | Numbers, Operations and Relationships |
| 2. Numbers (fractions)                      | Numbers, Operations and Relationships |
| 3. Numeric and geometric patterns           | Patterns, Functions and Algebra       |
| 4. Geometry and measurement                 | Space and Shape; Measurement          |
| 5. Other                                    |                                       |

Table 3: Mathematics Challenge categories and alignment to the curriculum

According to the classification all content areas of the Senior Phase, with the exception of Data Handling, were covered in the grade 7 paper. The writer had worked out a possible classification for the grade 7 paper, according to the categories shown in table 3. This classification is shown in table 4.

<b>Broad category</b>	<b>Number of questions</b>	<b>Actual questions</b>
1. Numbers and number sense (whole numbers)	5	1, 7, 8, 22, 23
2. Numbers (fractions)	4	2, 5, 6, 16
3. Numeric and geometric patterns	7	12, 13, 14, 19, 20, 21, 25
4. Geometry and measurement	8	3, 4, 9, 10, 11, 15, 17, 24
5. Other	1 (calculation of days). This question could have been classified under “measurement”.	8

Table 4: Possible classification of Mathematics Challenge questions (grade 7)

The writer does not claim that the classification given is the most definitive one. However, it does give a clear indication of the content covered in the paper. Students had to come up with their own classification of questions by working in groups. The groups were numbered group 1; group 2 and group 3. These classifications are shown in tables 5 – 7.

<b>Broad category</b>	<b>Number of questions</b>	<b>Actual questions</b>
1. Numbers and number sense (whole numbers)	2	1, 18
2. Numbers (fractions)	3	2, 5, 6
3. Numeric and geometric patterns	18	3, 4, 7, 9, 10, 11, 12, 13, 14, 15, 16, 17, 19, 21, 22, 23, 25
4. Geometry and measurement	1	24
5. Other	1	8

Table 5: Classification compiled by Group 1

<b>Broad category</b>	<b>Number of questions</b>	<b>Actual questions</b>
1. Numbers and number sense (whole numbers)	6	1, 7, 18, 19, 20, 21
2. Numbers (fractions)	4	2, 5, 6, 16
3. Numeric and geometric patterns	6	8, 12, 13, 14, 17, 25
4. Geometry and measurement	6	3, 4, 9, 10, 11, 15
5. Other	3	22, 23, 24

Table 6: Classification compiled by Group 2



Broad category	Number of questions	Actual questions
1. Numbers and number sense (whole numbers)	3	1, 7, 18
2. Numbers (fractions)	5	2, 5, 6, 10, 16
3. Numeric and geometric patterns	8	8, 12, 13, 14, 19, 20, 21, 25
4. Geometry and measurement	6	3, 4, 9, 11, 15, 17
5. Other	3	22, 23, 24

Table 7: Classification compiled by Group 3

### COMMENT ON THE CLASSIFICATION BY GROUPS

The writer did not expect the groups to provide the same classification as was expected. It would appear that students used their own experiences and intuition when classifying questions. Although the students were able to classify some of the questions in the paper correctly, they may not have had the necessary experience or background knowledge to classify all the questions. This prompted the writer to lead a class discussion on the classification of the questions to correct any misconceptions that students may have had. This was appreciated by the students.

### STUDENTS COMMENT ON THE TYPES OF QUESTIONS IN THE PAPER

The responses of students in their initial feedback did cover some comment on the type of questions in the paper and this was discussed earlier. In this activity, students had to discuss this matter in their groups. The students in group 1 stated that the questions were not easy. There were a lot of analyses involved and according to them the standard was very high for grade 7 learners. They complained that the questions were “complicated” and learners would become confused and frustrated when they did not know how to work out a problem.

Similar views were expressed by group 2. They commented on the “challenging” nature of the questions. Learners would spend too much time on the questions and probably get frustrated. Learners would then resort to guessing. These students complained about the phrasing of some of the questions claiming that this would be a deterrent to second-language learners. Learners who have not been exposed to these types of questions would find the questions difficult. This might lead them to think that they did not know anything.

Group 3 had a different view from the other two groups. They felt that the paper was not that difficult and that it was set at the right standard. They believed that the paper tested learners’ critical thinking and problem- solving skills and learners should have been taught the relevant content. Participation in the challenge would be a way of inculcating a love for mathematics.

The comments by the groups appeared to triangulate well with the individual feedback of the students stated earlier. This probably ensured that the data collected from the individual students and the groups were both valid and reliable.

### **INCORPORATING PROBLEM-SOLVING TYPE QUESTIONS INTO LESSONS**

Mathematics Olympiads or competitions should not exist as “stand-alone” events. Rather, problem- solving type questions should be incorporated into classroom lessons so learners are aware of problem- solving throughout the year, not only at Olympiad time. It was with this in mind that the groups were asked how they would incorporate problem-solving type questions into their lessons.

Group 1 stated that these problems could be given as projects or assignments and then be discussed in class. The teacher would provide solutions once these are handed in. These problems should be given more often so learners get used to these problems. The students in group 2 stated that class lessons should allow for the inclusion of one or two of these questions. These could be taken from past year papers and discussed in class. Learners should be taught the different problem-solving strategies so they could attempt these problems with confidence. These questions could also be given in homework. Group 3 expressed similar views to group 1 and 2. They believed that posters of geometric figures, number patterns and other content areas may help learners develop problem- solving skills.

### **PREPARING LEARNERS FOR MATHEMATICS OLYMPIADS OR COMPETITIONS**

Despite the majority of these students being new to Olympiads, the idea of this research was to point out the importance of Olympiads in the mathematics development of children. Thus, when they started teaching they should encourage their learners to participate in Mathematics Olympiads. Thus, as part of the group discussions, students were asked how they would prepare their learners for Mathematics Olympiads.

Group 1 emphasised the importance of reading and understanding. Learners would have to gain much needed practice by going through past year papers. They also stated that exposure to Mathematics Olympiads would help in developing “curiosity in learners”. Group 2 expressed similar sentiments to group 1. In addition, learners would need to know the basics and could work out problem-solving questions in groups. Group 3 also emphasised the need for practice and working in groups. By working in groups, learners would be able to share ideas and learn from each other. They could be given similar papers in class to determine their understanding of certain topics.



## PERFORMANCE IN THE FINAL ROUND

After the marking and analyses of the first round challenge paper, students were now more prepared for the final round. They wrote the final round approximately a month after the first round.

Table 8 shows the performance in the first round and final round.

Student	First Round	Final Round	Difference
A	9	22	+13
B	24	23	-1
C	4	24	+20
D	7	17	+10
E	6	19	+13
F	6	14	+8
G	7	15	+8
H	10	21	+11
I	13	19	+6
J	7	16	+9
K	19	21	+2
L	16	16	-
M	18	18	-
N	14	19	+5
<b>Average</b>	<b>11,43</b>	<b>18,85</b>	<b>7,42</b>

Table 8: Performance in the first and final round

## COMMENT ON FINAL ROUND PERFORMANCE

For 11 of the 14 students, performance in the final round (meant to be more difficult than the first round) improved. For one student there was a negligible drop of 1 while the other two student's marks remained the same. The overall average increased by 7, 42 marks. The performance in the second round is probably due to students paying more attention to problem-solving after the class activities where the questions were classified and analysed. The fact that the classroom activities also involved students discussing possible ways of incorporating these questions in their teaching and examining ways of preparing learners for Olympiads and Competitions may have also contributed to their improved performance in the final round.

## DISCUSSION

This research was prompted by a survey of the Mathematics Method students which revealed that only one of them had participated in a Mathematics Olympiad while at school. This was indeed a frightening statistic and did not bode well for any future learners taught by these teachers (once they qualified). It was highly likely that none of these students would expose their learners to Mathematics Competitions or Olympiads. The writer, a teacher trainer at the time, decided to intervene, starting at a more elementary level. In this regard, the South African Mathematics challenge was used as a vehicle to ascertain how the students approached problem-solving. Their performance in round 1 showed that these students definitely required more assistance with problem-solving.

In their feedback after writing round 1, most claimed that the paper was difficult and that they struggled with the paper. They had never expected it to be set at such a high standard for grade 7 learners. Clearly, their lack of experience in understanding and working through Olympiad type questions showed with their poor performance in round 1. This led to a structured intervention which the writer implemented during the Mathematics Method class periods. This intervention included marking the papers of about 900 learners, then dividing the students into groups and asking them to classify the questions in the grade 7 paper according to certain categories. They also had to comment on the types of questions in the paper, state how they would introduce problem-solving type questions into their lessons and indicate how they would prepare their learners for Mathematics Olympiads and Competitions.

The last part of the intervention involved the students writing the final round of the grade 7 paper of the South African Mathematics Challenge, about a month after round 1. It would appear that the intervention strategies were successful as there was a very substantial increase in their performance in the final round of the challenge. This is significant as the final round usually consisted of more difficult questions when compared to round 1.



## FINDINGS

This research was carried out in a structured manner with a view to helping the students develop their problem-solving abilities. These students had passed grade 12 mathematics with a minimum of 50% and had also completed all their mathematics 1 modules at the university. Thus, it would appear that they had the necessary knowledge and skills for problem-solving and the intervention was designed to bring these abilities to the fore.

It is now possible, using the data from this research and the discussion in the previous section, to come up the findings in this research. This would be written within the context of the research sub-questions and the research question.

### **How do pre-service students perform in mathematics competitions which target their learners?**

It is evident from the performance in the first round of the South African Mathematics Challenge that these students performed poorly. They provided various reasons for this performance. Despite these students being good mathematics students at various levels, their lack of experience in working with Olympiad- type questions probably caused this poor performance. However, this was done before the structured intervention.

### **Can the performance of pre-service students in mathematics competitions be improved through a structured intervention?**

The strategy for the structured intervention consisted of students giving initial feedback, marking papers, classifying the grade 7 questions, commenting on the types of questions, stating how they would incorporate such questions in their lessons, and saying how they would prepare learners for Mathematics Olympiads. This strategy appeared to change the way these students viewed problem-solving, with them becoming more confident and positive.

This was clearly evident when they wrote the final round of the Mathematics Challenge, with nearly all students improving their marks. The improved performance of students, especially in a more difficult paper, would probably point to the structured intervention being classified as successful.

## **What are the views of pre-service students on the mathematics competitions such as the Mathematics Challenge?**

Despite the initial poor performance of the students and their views that the Mathematics Challenge was difficult for primary school learners, there was agreement that the standard of teaching and learning mathematics in primary schools was not at the level it should be. In this regard, learners would not be able to confidently participate in competitions such as the Mathematics Challenge. However, raising the standard of teaching and learning in the primary schools would make learners better equipped for Mathematics Competitions.

Unlike their own lack of experience with regard to participation in Mathematics Olympiads, these students believed that all learners should be given the opportunity of participating in competitions such as the Mathematics Challenge. This would help the learners' mathematical development. With the current focus by the Department of Basic Education on improving teaching and learning in the primary schools through the introduction of Annual National Assessment (ANA) in mathematics and language, it would be interesting to compare learner performance in the Mathematics Challenge and their performance in the ANAs.

## **Can pre-service students be encouraged to incorporate Olympiad type problems in their classrooms?**

The data in this research suggests that the response to this question would be in the affirmative. These students were able to outline a number of ways in which they could incorporate Olympiad-type problems in their classrooms. They realised the importance of these problems in helping with children's mathematical development.

These students also indicated how they would help their own learners prepare for Mathematics competitions. This is an important breakthrough in their own training as it showed that these students were likely to introduce their learners to Mathematics competitions when they started teaching. This would be a far cry from their own lack exposure when they, themselves, were at school.

## **The research question: Are pre-service teachers of mathematics competent in solving Olympiad-type problems?"**

An interrogation of the marks from round 1 of the Mathematics Challenge suggests these students were not competent in solving Olympiad-type problems. This lack of competence was probably due to a lack of experience in working with these problems.

However, other data from this research showed that this lack of competence could be changed through a structured intervention. This structured intervention made the students aware of the need to expose their learners to Mathematics Competitions and Olympiads. This could only be done if they themselves were competent at solving these types of problems.



## CONCLUSION

This study was conducted with a group of 14 pre-service mathematics students. 13 of these students had never taken part in a Mathematics Olympiad or Competition before. In line with the theories on experiential learning, these students would not have been able to pass on these experiences to their own learners. The writer attempted to address this with a structured intervention, using the South African Mathematics Challenge as a vehicle. This intervention was able to turn things around. The students became well-versed in problem-solving and problem-solving strategies. There is no doubt that these students would be able to promote Mathematics Olympiads and Competitions at their schools.

This study has great significance on the future of Mathematics Olympiads and Competitions in South Africa. The majority of our mathematics teachers have had little or no exposure to Mathematics Olympiads and Competitions. It is more than likely that their learners have not been exposed as well. This may mean that the majority of learners are not given an opportunity to enrich their mathematics knowledge and skills. The structured intervention described in this study provides a possible way in which this could be addressed.

## REFERENCES

- Cohen, L. and Manion, L. (1985). *Research methods in education*. 2<sup>nd</sup> ed. London: Croom Helm.
- Department of Basic Education (2011). National Curriculum Statement (NCS). Curriculum and Assessment Policy Statement. CAPS. Senior Phase Grades 7-9. Government printing works. Pretoria
- Department of Basic Education (2013). Ministerial task team: Investigation into the implementation of Maths, Science and Technology (24 October 2013).
- Hatch, J. A. (2002). *Doing qualitative research in education settings*. Albany: State University of New York Press.
- Haynes, C. (2007). *Experiential learning: Learning by doing*. Retrieved February 18, 2014 from [http://adulthoodeducation.wikibook.us/index.php?title=Experiential\\_Learning\\_-\\_Learning\\_by\\_Doing](http://adulthoodeducation.wikibook.us/index.php?title=Experiential_Learning_-_Learning_by_Doing)
- Skowronekand, D & Duerr, L. (2009). The convenience of non-probability, Survey strategies for small academic libraries. *College & Research library news*. Vol. 70 (7), 412 - 415
- The South African Mathematics Challenge, (2014). Retrieved February 24, 2014 from <http://www.amesa.org.za/Challenge/Index.htm>

## EXPLORING THE MATHEMATICAL PROFICIENCY OF GRADE 6 TEACHERS: A CASE OF GAUTENG TSHWANE EAST

Zingiswa M. Jojo, Joseph J. Dhlamini, M.M. Phoshoko and M.G. Ngoepe

Department of Mathematics Education, University of South Africa

*As part of the community engagement initiative, the Department of Mathematics Education (DME), at the University of South Africa (UNISA), conducted a pilot study to explore the mathematics proficiency of teachers in Gauteng Tshwane East district. Six Grade 6 mathematics teachers from 11 schools participated in the study. We conceptualize proficiency as a multi-faceted notion having dimensions of teacher knowledge, instructional practices, assessment practices and contextual factors. We adopted a mixed-methods approach consisting of exploratory and survey designs with both qualitative and quantitative approaches. For triangulation, three instruments were used, namely, teacher questionnaire, teacher interviews as well as classroom observation schedule. The results of the study suggested that study participants are proficient in terms of demonstrating the following aspects of classroom instruction: mathematics knowledge, appropriate assessment techniques; ability to handle educational and socio-economic challenges. Further analysis of data revealed that teachers' proficiency is however downplayed by a persistent emergence of unintended issues relating to educational and socio-economic challenges. In turn, because the educational agenda is altered teachers fall short in realizing the full potential of their teaching proficiency. We therefore recommend that the unintended issues should be addressed in order to lay suitable ground for effective instruction. We further acknowledge that the insights gained from this pilot should assist the DME to design training programs to equip mathematics teachers to mitigate the influence of the unintended classroom agenda.*

**Keywords:** Mathematical proficiency; teacher knowledge; instructional practices; assessment, and mathematics teachers

### INTRODUCTION

The poor performance in mathematics has continued to become a great source of worry in many countries around the world. In an attempt to address this problem South Africa has participated in several evaluative and comparative studies that were conducted at national, continental and international levels, in order to identify the actual status of the problem. Results from these participations consistently point to a serious problem to the quality of mathematics instruction in South Africa.



In 2000 and 2007, Grade 6 learners in South Africa participated in the Southern and East African Consortium for Monitoring Educational Quality (SACMEQ) studies, and were tested in mathematics (numeracy) and reading (literacy). In the 2000 study (SACMEQ II) around 80% of South African Grade 6 learners that participated reached the lower half of eight levels of competence in mathematics on the SACMEQ continuum (Moloi & Strauss, 2005). The results of both 2000 and 2007 participation by South African learners are shown in Table 1. The results in Table 1.1 show that the mathematics performance of Grade 6 learners in South Africa is not very good. In 2000, three provinces scored above the SACMEQ II mean average of 500. However, the country’s average mean was 486. Again, in 2007 South Africa failed to reach the SACMEQ III average of 510. The SACMEQ (2013) has observed that although South Africa has made significant strides taken to transform the nation from pre-1994 apartheid system, the main challenge that remains almost intact is improving the levels and quality of educational outcomes as measured by the reading and mathematics, Rasch score obtained by the Grade 6 learners in SACMEQ tests.

**Table 1: South Africa’s performance in the reading and mathematics in the SACMEQ II and SACMEQ III**

	Pupil reading score		Pupil mathematics score	
	2000	2007	2000	2007
Eastern Cape	444	448 ▶	449	469 ▲
Free State	446	491 ▲	448	492 ▲
Gauteng	576	573 ▶	552	545 ▶
KwaZulu-Natal	517	486 ▼	510	485 ▼
Mpumalanga	428	474 ▲	433	476 ▲
Northern Cape	470	506 ▲	461	499 ▲
Limpopo	437	425 ▼	446	447 ▶
North West	428	506 ▲	420	503 ▲
Western Cape	629	583 ▼	591	566 ▼
<b>SOUTH AFRICA</b>	<b>492</b>	<b>495</b> ▶	<b>486</b>	<b>495</b> ▶
<b>SACMEQ</b>	<b>500</b>	<b>512</b> ▲	<b>500</b>	<b>510</b> ▶

Values in **Green** = 10 points or more above SACMEQ II mean of 500

Values in **Black** = less than 10 points above or below SACMEQ II mean of 500

Values in **Red** = 10 points or more below SACMEQ II mean of 500

Notes about trend:

▲ Increased by 10 points or more

▶ Minimal change (less than =10)

▼ Decreased by 10 points or more

*Source: SACMEQ (2013)*

Some researchers believe that the remnants of apartheid play a significant role in the teaching and learning of mathematics in South African schools. The rationale for this line of thinking is based on the type of teachers who were trained under the apartheid system.

These teachers have been identified at primary level in mathematics (Howie, 2001; Moloi & Strauss, 2005) and at secondary level in mathematics (Howie, 2001) as unqualified and under qualified. This has continually resulted in an increasing poor Grade 12 output and the majority of learners who do not meet the minimum requirements for university entrance (Modisaotsile, 2012).

Although factors such as overcrowded classrooms, high dropout rate, low literacy and numeracy, snail's pace of teachers progress through curriculum, lack of resources, and ineffective leadership management, have been identified, poor teacher training, unskilled teachers, lack of commitment to teach by teachers and a shortage of resources in education seem to be the most determinant of learners' poor performance in mathematics. We believe the latter factors are closely related to the notion of proficiency that a teacher is supposed to reflect at classroom level. This paper reports on a study that explored the teaching proficiency of Grade 6 mathematics teachers, and primarily taking into cognisance the existence and emergence of other factors that have a direct bearing on the teaching and learning process.

### **AIM OF THE STUDY**

This study was aimed at identifying teachers' proficiency in the teaching and learning of mathematics in primary schools. To achieve this aim, researchers set out the following objectives for the study:

- To evaluate the teachers' mathematical knowledge for teaching;
- To determine the teacher's ability to interact effectively with learners in order to promote meaningful learning; and,
- To ascertain teachers' ability to deal with learners' educational and socio-economic factors that may affect performance in mathematics.

### **CONCEPTUAL FRAMEWORK**

Studies that have explored the notion of proficiency have largely focussed on learners (Kilpatrick, Swafford, & Findell, 2001). The study that is reported in this paper offers a unique dimension in that its focus is on the teaching proficiency of mathematics teachers. With regard to teachers, we conceptualize proficiency as a multi-faceted, multi-layered notion hence the dimensions of proficient teaching that we reflect in this paper are: (i) teacher's content knowledge, (ii) teacher's instructional practices within a well-defined classroom environment, (iii) teacher's use of assessment techniques to complement and develop the lesson, and (iv) teacher's response to contextual factors that impact on the development of learners.



Thus in short, we define a proficient teacher is defined as one who has: (i) mathematical knowledge for teaching, (ii) background awareness of the learners (sensitive to the social variables, environmental factors), (iii) abilities to interact effectively with learners, (iv) reflects meaningful habits of reflection towards the teaching practice, and (v) demonstrate understanding of teaching that is situated in a problem solving environment.

Among all the dimensions of proficient teaching that we present in this paper, we regard teacher's subject matter knowledge (SMK) as being the most significant and fundamental variable. This thinking is in line with the view that the quality of instruction that teachers provide for the learners is largely influenced by the quality of knowledge the teachers possess since teacher subject matter knowledge of mathematics, for instance, is fundamental to teacher ability to provide effective mathematics teaching (Ball, Hill, & Bass, 2005; Kreber, 2002). Therefore, the ineffectiveness of the mathematics teachers or the teaching proficiency of these teachers could be a major determinant for the learners' performance in mathematics.

## OUR STUDY

### Research design

This study adopted a mixed-methods approach consisting of exploratory and survey research designs. The survey research design informed the researchers what the teachers' level of mathematical knowledge for teaching is, and what strategies teachers use to address the confounding issues of educational and socio-economic origin. In addition, the exploratory research helped to determine the current status of the teachers' proficiency in mathematics. This paper reports specifically on the data collected for pilot purposes in order to initially familiarise researchers with the teaching proficiency of mathematics. The results of this exploration will later inform an envisaged intervention strategy for the improvement of teachers' proficiency in mathematics.

### Population and sample

The population for the study comprised of Grade 6 mathematics teachers in the intermediate phase (Grade 4 to Grade 6) in the Gauteng<sup>1</sup> Tshwane district. From this population 11 primary schools participated in the study. Grade 6 is purposefully chosen because it is part of the action plan 2014 towards the realisation of schooling 2025 (Department of Basic Education [DBE] 2012: 8), which places Grade 6 as one of the focal grade levels. From the 11 primary schools six teachers participated in the study.

A convenience sampling technique was utilized to select schools that will provide easy access and maximization of data collection for the study.

---

<sup>1</sup> Gauteng is one of the nine provinces of South Africa.

Permission to conduct this research was sought and acquired from the Gauteng Department of Education as well as principal and Grade 6 teachers of the respective schools.

### **Instrumentation**

The three instruments that were used in this study were administered on six Grade 6 mathematics teachers, and these were lesson observation schedule, interviews and questionnaires. The questionnaires explored teachers' demographic details (section B) and issues relating to classroom practice (section B). Some of the issues addressed in Section B were teaching practices, teaching material and curriculum implementation. The interview schedule explored the following issues: (i) mathematical knowledge for teaching, (ii) instructional practices, (iii) assessment practices, and, (iv) educational and socio-economic factors. Some of the issues that were addressed by the lesson observation included: the availability of the lesson plan, teacher-learner interaction, demonstration of teachers' knowledge, use of assessment techniques as well as the availability of resources. Given that this study took the format of a pilot exercise, the assessment of measurement properties of each instrument would then be addressed during the course of data analysis.

### **Data analysis**

Both qualitative and quantitative methods of data analysis were used.

### ***Questionnaire***

Data from the teachers' questionnaire show that of the six teachers only one has a degree qualification (BEd) in mathematics, while four were in possession of a three year senior primary teachers' diploma, and one had an Advanced Certificate in Education (ACE). Furthermore, it was noted that the level and depth of content knowledge in most of the qualifications that teachers possessed did not exceed that of the learners. Of the 11 teachers only two were studying further and one of them was studying towards school management. We observe that studies that are geared towards a management direction do not necessarily empower teachers to enhance their teaching proficiency as management studies are primarily appropriated towards school management and leadership issues. The qualifications of teachers (n=6) are represented in Table 1.



**Table 2: The qualifications of teachers who participated in the study (n=6)**

Types of teaching qualification	Highest qualification in Mathematics	Further studies
ACE, n=1	Less than Grade 12, n=1	No, n =4
SPTD <sup>2</sup> , n=4	Grade 12, n=2	
BEd <sup>3</sup> Hons, n=1	More than Grade 12, n=3	Yes, n=2

In addition to the teachers' qualifications in Table 1, we also found that four teachers had more than 10years experience of teaching Grade 6 mathematics, while one had less than 6years and the other had just started teaching. The latter had previously taught high school mathematics but joined the intermediate phase for promotional purposes. Two teachers indicated that their classes were overcrowded while others reported that their classes were manageable at 43 and 45. With regard to the level of readiness to implement the Curriculum and Assessment Policy Statement (CAPS), only one teacher indicated that she was 'completely ready' while two teachers indicated to be 'almost ready' and three were 'slight ready'.

### **Teacher interviews**

The interviews were meant to address the following themes:

#### *Mathematical knowledge for teaching*

Five teachers indicated that the CAPS training they received for mathematics teaching in Grade 6 was inadequate since it was run in a period that was less than a week. For instance, one teacher said:

*'Yes, when I did PTC and diploma (previous Vista University) in mathematics, the training was enough. But knowledge changes every day, and one needs to keep abreast. Learning is lifelong.'*

In an attempt to improve her teaching proficiency this teacher has made special effort to join the Association of Mathematics Educators of South Africa (AMESA) for the enrichment of mathematics teaching. She further indicated that she attends all workshops, especially those that are content related. Generally, teachers indicated that they receive training and attend workshops that are organised by the Department of Basic Education (DBE), the Non-Governmental Organisations (NGO's), Higher Education Institutions and those arranged by mathematics education associations.

<sup>2</sup> SPTD is an acronym for Senior Primary Teachers' Diploma.

<sup>3</sup> BEd stands for Bachelor of Education.

As a follow up, teachers were further asked to reflect on the extent to which these trainings and workshops impacted on their teaching of mathematics. All teachers (n=6) indicated that the workshops/ training were helpful as captured in the following extracts:

*“These workshops are helpful and open doors for us to see where we lack knowledge and how to close the gap”.*

*‘It clarified a lot of topics that I did not understand in mathematics’*

#### *Instructional practices*

Teachers reported teaching practices that ranged from the old traditional (conventional) methods, such as telling method/ information session, and textbook method, to the reform approaches, which represented group work, examples method, questioning skills, investigations, and problem solving. Teachers who favoured the group approach mentioned that this teaching method is facilitated by grouping learners heterogeneously. The other teacher emphasized that learners’ opinions were valued in her lesson. Other teachers indicated that using real-life examples them to facilitate the understanding of the mathematical concepts by moving from the known to the unknown (from concrete to abstract). It also appeared that teachers use different yardsticks to measure the effectiveness of the method of teaching used in mathematics. Some teachers measured the effectiveness of their instruction in terms of learner performance, while others used learner participation (classroom interactions) as a measure. For example, one teacher reported that group work led to the learners sharing ideas and demonstrating improved participation and communication. Another teacher said:

*‘Yes, there is 100% pass rate.’* And she continued to say,

*‘If the pass rate drops, then the method is not effective.’*

*Learners learn from others in the group, getting explanations about the project from others.*

With regard to learners’ learning difficulties teachers said the identified them through assessment activities such as class test, non-participation in activities, misinterpretation of questions, language challenges as a barrier and sometimes when the learners find it difficult to do mental mathematics.



They further reported that learning difficulties would then be rectified by using mother tongue to explain the concept (code-switching), re-assessment, and initiating remedial programs and extra classes either in the afternoon or in the morning, but teachers insisted that the afternoon classes were mostly impossible carry out since the learners use common transport and are to leave immediately after school.

Another teacher reported that her struggling learners are referred to the School Based Support Team (SBST), which deals specifically with learners who experience challenges during learning. The SBST units are subject-specific.

#### *Assessment methods used to facilitate learning*

Tests, assignments, projects, investigations, examinations, homework, classwork and research were some of the assessment methods suggested to be useful during mathematics lesson. These assessment techniques helped the teachers to improve their practice, to improve and identify learners who have passion for mathematics. The classwork (CW) and the homework (HW) received a popular approval from the teachers (n=6). Teachers insisted the two forms of assessment, CW and HW, were informal in nature but are useful to provide a formidable preceding foundation for the formal assessments such as tests, assessments, assignments and projects. In particular, classwork was chosen as the most beneficial method since it was mentioned to be a tool to measure the effectiveness of all other assessment methods. CW can be used to identify learners' problems early before the test. Homework was indicated by one teacher to be the one done after more understanding and helps the learners to revisit the work done. On the average teachers rated the significance of assessment as an indicator of the efficiency of the teaching method. Assessment helps to design a responsive instruction and could help the teacher to do a self-reflection to improve the lesson instruction.

#### *Educational and socio-economic influences on the learning of mathematics*

Socio-economic factors that teachers highlighted included learners' challenges of poverty, learners' places of dwelling, which in this context was shacks or informal settlement that is largely without electricity and other basic services, the fact that learners do not do homework, poor concentration in class, unavailability of school uniform that lead to the affected learners being given by the school the uniform from previous learners (uniform exchange), child-headed families, learners staying with grandparents who are largely illiterate and unable to help children with homework. The educational factors included that emanated during data collection included learners with learning barriers and those who were described by teachers as having 'very low IQ'.

The teachers revealed that learners who are affected by both socio-economic and educational barriers are identified through the following methods and behaviours: non-participation in class, withdrawal from classroom activities, being an orphan, visitations to their homes, learners who are restless in class, learners who are always tired, learners who always visit the sickroom, and at times asking learners directly and personalized learners.

Uniform exchange, involvement of social workers and social care initiatives, home visits and the feeding scheme (a program where needy learners are provided with food at school) were indicated as preferred methods of addressing the educational and socio-economic factors that affect the learning process. Sometimes these challenges are addressed by the School Management Team (SMT), and parents are also invited to the school. Teachers describe the impact of these factors as having a negative influence on: (i) the pace at which the teacher are supposed to be teaching, (ii) the focus and concentration of learners in class, (iii) the quality of work produced by the affected learner,, (iv) the extent of interaction that the learner enjoys with fellow learners, (v) the self-esteem of the learner. The other point that was highlighted by teachers was that learners who are adversely affected by socio-economic and educational challenges end up losing interest in the educational agenda. These learners eventually become motivated to go to school by the fact that they get food and that the transport is provided from them to go to school.

### ***Classroom observation schedule***

The classroom observations looked at the availability of the lesson plan, approaches used to introduce the lesson, general class management, teacher interaction with learners, teachers' knowledge, the use of chalkboard, as well as assessment strategies that characterize instruction. All the teachers that participated in this study were part of GPLMS<sup>4</sup> program that which was launched for primary schools by the Gauteng Department of Education. This arrangement made it possible for all teachers (n=6) to have lesson plans. However, as researchers we wanted to look at how the teachers articulated the lesson plan, hence we conducted lesson observations to inform our judgement in this regard.

---

<sup>4</sup> GPLMS refers to Gauteng Primary Language and Mathematics Strategy, which is a special intervention program that was launched by the Gauteng department of Education to improve learners' performance.



Upon our interaction with teachers, they registered a concern that a template given to them as a work schedule deprived them the freedom that would allow them to manifest their teaching proficiencies. They mentioned that the templates prescribe the programme of activities for teachers, which at times fail to address immediate educational needs for learners. For example, the mental mathematics activity for learners is allocated 6 minutes and this arrangement seems not to allow learners to digest accordingly the conceptual material embedded in this learning activity. Therefore teachers are seemingly not happy with the DBE arrangement and prescriptions as they limit their innovative abilities. They felt their professional competences are undermined.

This disadvantages the learners since the teacher has to move to the next concept even if the teacher realizes that learners have not understood the concept. Therefore at times the time allocated to teach certain concept is insufficient. For example, the Grade 6 schedule for mathematics prescribes that fractions should be taught within two hours. In this regard one teacher noted:

*'...but because you want to comply, you need to move to the next topic even if you have not exhausted the previous one. You just have to touch on a certain topic so as to have evidence in the learners' book to show the facilitator.'*

The preceding comment by the teacher highlights some of the variable that impact negatively to the nourishment of the proficiency of teachers in Grade 6. Teachers opt to comply, as suggested by the comment, and subsequently hold back their inputs to the teaching and learning process. Using observations from the tool of lesson observation, researchers felt that teachers demonstrated proficiency in the following areas of exploration:

(i) *Interaction with their learners*

During a lesson presentation most teachers demonstrated familiarity with the use of reformed methods of teaching, such as allowing the lesson to be learner-centred. Other teachers were observed to be moving around in class in order to provide individual assistance to learners. Also, learners were readily posing questions to the teachers from time to time.

(ii) *Mathematical knowledge*

Our observation as researchers reflected that teachers were adequately knowledgeable in their subject matter. This was demonstrated when teachers satisfactorily provided useful explanation of concepts they were teaching. The use of real-life examples characterised the facilitation and the development of the lesson.

### (iii) *Usage of chalkboard*

Not all teachers made use of the chalkboard during observation. However the few teachers that used this teaching equipment were observed to be efficient. When writing on the chalkboard the work was partitioned into sections and the handwriting was legible. In some instances learners were called to come in front to represent their responses on the chalkboard.

### (iv) *Demonstration of assessment techniques*

During lesson observations, the following assessment strategies were observed: Peer assessment, usage of mental mathematics activity, oral questioning, classwork and homework.

## **DISCUSSIONS AND CONCLUSIONS**

Given the reported activities of this exploratory study, the research was launched in participating schools with an aim to explore the levels of proficiency of mathematics teachers in Grade 6. The results of this study reveal that the nourishment and effective growth of teachers' proficiency is downplayed by variables such as those relating to socioeconomic factors and language demands. The lesson observations, which were conducted on teachers seem to suggest that teachers have the required level of mathematics proficiency that is needed to teach Grade 6 learners certain concepts of mathematics. However, challenges that persistently play themselves out during the course of instruction create an atmosphere that makes it possible for teachers to adequately exhibit pedagogical qualities of proficiency. For instance, most learners who are taught by these teachers experience pressing challenges that eventually detract their focus from the intended educational agenda, and this aspect impacts negatively to the manifestation of the qualities of proficiency of the teacher.

In essence, the results of this study help us to realize that the conditions in the selected schools are not educationally conducive to allow teachers to manifest aspects of proficiency in their subject. The emergence of issues of poverty continues to confront teachers in a manner that tragically defeats the agenda of teaching and learning. Instead of grounding themselves with issues of teaching and learning, which should nourish their instructional proficiency, teachers have to deal with issues of hunger, abuse and low esteem that are entrenched in the daily lives of their learners. This circle of realistic events contributes negatively to the teaching and learning agenda. As mentioned in the analysis, in almost all schools, teachers were using GDE developed lesson plans (GPLMS), which they are expected to follow to the latter. Researchers observed a disjuncture with regards to the implementation of the lessons plans versus their proficiency. The fact that the already-prepared lessons do not create space for teachers to



navigate freely through the syllabus; it seems the opportunities for teachers to strengthen their teaching proficiencies are subsequently diluted.

We therefore recommend that the confounding influence of unintended factors that continue to play themselves in mathematics classrooms should be mitigated effectively in order to facilitate teaching and learning in our schools.

## REFERENCES

- Ball, D. L., Hill, H. C., & Bass, H. (2005) knowing mathematics for teaching: who knows mathematics well enough to teach third grade, and how can we decide? *American Educator*, 14-22.
- Cohen, D. K., & Hill, H. C. (2001). *Learning policy: When state education reform works*. New Haven, CT: Yale University Press.
- Community agency for social enquiry (CASE) and JET Education services (JET) CASE/JET, (2007). *Learner absenteeism in the South African schooling system*. Commissioned by the Department of Education. Johannesburg: JET. Available at <http://www.jet.org.za/item.php?Id=265>.
- Department of Basic Education. (2012). *Report on the Annual National Assessments 2012: Grades 1 to 6 & 9*. Government Printing Works.
- Department of Education. (2005a). *National report on systematic evaluation: Grade 6, Intermediate Phase*. Pretoria.
- Department of Education. (2005b). *Intermediate phase: Systematic evaluation report, grade 6*. Pretoria.
- Desimone, L. M. (2009). Improving impact studies of teachers' professional development: Towards better conceptualizations and measures. *Educational Research*, 38, 181-199.
- Desimone, L. M., Porter, A. C., Garet, M., Yoon, K. S., & Birman, B. (2002). Does professional development change teachers' instruction? Results from a three-year study. *Educational Evaluation and Policy Analysis*, 24(2), 81-112.
- Dieltiens, V. (2006). Systematic evaluation and the national protocol on assessment. *Quarterly Review of Education and Training in South Africa*, 13(1), 5-6.
- Garet, M. S., Porter, A. C., Desimone, L. M., Birman, B., & Yoon, K. S. (2001). What makes professional development effective? Analysis of a national sample of teachers. *American Educational Research Journal*, 38(3), 915-945.
- Howie, S. (2001). *Mathematics and science performance in Grade 8 in South Africa 1998/1999*. Pretoria: Human Sciences Research Council.
- Howie, S. (2004). A national assessment in mathematics within an international comparative assessment. *Perspectives in Education*, 22, 149-162.
- Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.
- Kreber, C. (2002). Teaching excellence, teaching expertise and the scholarship of teaching. *Innovative Higher education*, 27 (1), 5 – 23.
- Loucks-Horsley, S., Hewson, P. W., Love, N., & Stiles, K. (1998). *Designing professional development for teachers of science and mathematics*. Thousand Oaks, CA: Corwin Press.
- Modisaotile, B. M. (2012). *The Failing Standard of Basic Education in South Africa* <http://www.ai.org.za/wp-content/uploads/downloads/2012/03/No.-72.The-Failing-Standard-of-Basic-Education-in-South-Africa1.pdf>

- Moloi M. & Strauss, J ( 2005). A study of the conditions of schooling and the quality of education. South Africa working report. Harare and Pretoria: SACMEQ and the Ministry of Education.
- Moloi, M., & Strauss, J. (2005). The SACMEQ II project in South Africa: A study of the conditions of schooling and the quality of education. Harare: SACMEQ.
- Taylor, N. (2009). The state of South African schools Part 1: Time and the regulation of consciousness, *Journal of Education*, No. 46, 2009.



## LEARNERS' EXPLANATIONS OF THE ERRORS THEY MAKE IN INTRODUCTORY ALGEBRA

Sbongile Mashazi

Jules High School & Wits Maths Connect Secondary Project, School of Education,  
University of Witwatersrand

*This paper explores the thinking underlying Grade 9 learner errors in introductory algebra. The research used qualitative methods composed of documentary analysis as well as task-based interviews. Data was analysed using Kuchemann's (1981) six interpretations of letters in algebra. The results indicate that various aspects contribute to learners' errors including task instructions, new knowledge, ignoring of the letter, and replacing letters with numeric values.*

### INTRODUCTION

I begin this paper by giving the background to the study. I then briefly provide the conceptual framework and literature review about the research study. Thereafter I elaborate on the methods used as well as the participants in the study. I then describe how the data was collected. This is then followed by data analysis and discussion of findings.

### BACKGROUND TO THE STUDY

Prior to this research project, I was ignorant of the errors and misconceptions that learners produce in algebra and could not address these in my teaching. The problem of low learner achievement in South Africa does not seem to be subsiding (Simkins, 2013). Moreover, the Annual National Assessment (ANA) of 2011 revealed that, "the overall performance of learners was very low with average scores of 30%" (DBE, 2011, p. 2). These results also indicated that "domains in which learners displayed most serious weaknesses included patterns and mathematical functions" (DBE, 2011, p. 33). In addition, poor performance in higher grades (9-12) was linked to poor performance in algebra.

According to the RNCS "algebra is the language for investigating and communicating most of Mathematics and it can be seen as generalised arithmetic, extended to the study of functions and other relationships between variables" (DoE, 2002, p. 62). Algebra plays an important role in high school and tertiary education. Understanding why learners produce errors in algebra might be the beginning of the solution to low performance in mathematics since it may enable teachers to identify difficulties and obstacles that learners encounter when developing algebraic concepts. Hence, attention needs to be given to helping teachers to teach algebra in a meaningful way where they would address and rectify the usual errors that learners produce. I believe that learners and our education system in general stand to benefit from the findings of this research.

## PURPOSE OF THE RESEARCH

The main purpose of the research project was to explore learner thinking underlying errors that learners produce in introductory algebra. This was done by looking into how Grade 9 learners interpret letters in different algebraic tasks. The research project was guided by the following question:

- What are learners' explanations behind their responses to introductory algebraic tasks?

## LITERATURE REVIEW

According to the Oxford dictionary, algebra is a part of mathematics that uses letters and other symbols to represent quantities and situations. Furthermore, algebra is one of the most important topics in mathematics that develop learners' problem solving and analytical thinking skills (Schoenfeld, 2007). Hence, learner thinking is of great importance in this study. I would define learner thinking as the process that occurs in learner's mind when applying the existing knowledge to solve a certain task.

According to Olivier (1989), the existing knowledge is structured in a learner's mind into interrelated concepts called schema. These schemas are important tools that can be retrieved and used by learners when they encounter a similar scenario. If the new knowledge is not connected to the existing schema, that knowledge becomes isolated, which results in learners memorizing it in a form of rote learning (Olivier, 1989).

Hence, "errors and misconception are the natural results of learners' effort to construct their own knowledge" (Olivier, 1989, p. 13).

Large-scale research in countries such as Britain (Kuchemann, 1981), Australia (Stacey & MacGregor, 1994 & 1997) and others has shown that most learners encounter learning difficulties in understanding algebra and this has manifested in the range of errors and misconceptions, which are a characteristic of learners' responses. According to Christou, Vosniadou & Vamvakossi, the tendency of learners "to use their prior experience with numbers in the context of arithmetic" (2007, p. 289) interferes with the interpretation of letters in algebra. Again, learners have a tendency of simplifying an algebraic answer to a single term. Booth (1999) and Stacey & MacGregor (1994) refer to this error as conjoining of algebraic terms. I have witnessed these kinds of errors in my classroom. According to Stacey & MacGregor (1997), these errors have to do with the stages of cognitive growth required for an individual to progress from concrete to abstract reasoning.



## CONCEPTUAL FRAMEWORK

My research project was informed by Kuchemann's (1981) learner interpretation of letters in algebra. According to Kuchemann learners interpret letters in six different ways: letter evaluated, letter not used, letter used as an object, letter used as a specific unknown, letter as generalized number and letter used as a variable. He further highlights that ascribing different meaning to letters by learners in algebra determines the difficulty of a problem and the extent to which a learner would engage with algebraic problems. In this paper, I focus only on 'letter evaluated', 'letter not used', 'letter used as an object' and 'letter used as an unknown'. The description of these letter interpretations is given briefly below, making use of examples drawn from the literature.

*Letter evaluated:* It refers to problems that require learners to find the value of an unknown without actually operating on that specific unknown. For example finding the value of  $y = x - 3$  if  $x = 6$  or calculating the value of  $p$  if  $p + 2 = 7$ . In the first example, learners could evaluate the expression by substituting a given value for  $x$  to obtain the value of  $y$ . In the second example, learners might view a letter as missing information and this can lead learners to solve the problem by inspection. Booth (1999) refers to this way of solving as an informal method, which might lead to some learners failing to solve similar problems that would typically be solved by transposition or by working with additive identities, such as "find the value of  $p$  if  $p + 2 = 7 + 2p$ ". Of course it is also possible, but unlikely, that a learner could solve the above equation by reasoning that  $p + 2 = 5 + 2 + p + p$  and concluding that  $p = -5$ .

*Letter not used:* In some questions, learners can succeed without actually using the letter, hence the description 'letter not used'. In this case the letter is replaced by a given value. For example, "calculate the value of  $2(x + y) - 4$  if  $x + y = 15$ ". In such questions learners calculate an answer by substituting the given value. It is noted that learners acknowledge the existence of the variables in a question but do not make sense of them since the answer is a numeric value. Booth (1999) sees such problems as arithmetic since the aim is to find a numeric answer.

*Letter used as an object:* A variable is treated as shorthand for an object where it represents a length or size of a figure. For example, calculate the perimeter of a shape with equal length of 3 units if there are  $n$  number of sides. Again, it could be regarded as an object when matching or grouping the like-terms. For example, simplify

$2b - y + 3b$ . In this instance, the meaning of a variable is reduced from abstract to a concrete situation. According to Booth (1999), the unknown tends to have a different meaning when learners view  $5b$  as 5 bananas instead of 5 times the number of bananas.

*Letter as a specific unknown:* It refers to learners viewing a letter as an unknown such that they accept an algebraic expression as an answer. For example, find the product of  $3x(2x + 4)$  or add  $7$  to  $x + 1$ . In this instance, the letter is considered as having a specific value, which is not known at that time. Hence, the final answer would be in terms of a variable.

The literature review and conceptual framework guided me to understand the errors and misconceptions that learners produce and gave me an indication of how learners interpret letters.

## METHODS AND PARTICIPANTS

The participants in the study were Grade 9 learners with ages ranging between 13 and 16 years. I analysed 30 Grade 9 test scripts collected in October 2011 by the Wits Maths Connect Secondary (WMCS) project at a school in the project, to understand errors that learners produce in algebra. After noticing the common errors I developed task-based interviews which I administered with six Grade 9 learners in May 2012. The criterion for selecting these learners was based on their performance in 2012 term 1 mathematics results. Learners who obtained between 45% and 60% qualified to participate in the study.

## DATA ANALYSIS AND DISCUSSION OF FINDINGS

As I mentioned, I analysed 30 scripts and conducted task-based interviews with 6 learners, I discuss these two separately and provide some samples of learners' responses.

### Documentary analysis of test scripts

The analysis of this data gave me insight into the kinds of errors, mistakes and misconceptions that learners are likely to produce in introductory algebra. Furthermore, it assisted me to develop the instrument for the task-based interview. In one of the questions, learners were asked to "Multiply  $n+5$  by  $4$ ". This question was aimed at exploring the use of brackets when multiplying an algebraic expression with two terms. All learners responded to this question and the range of their responses is illustrated in figure 1 below. About 10% of learners responded correctly ( $4n+20$ ) suggesting that they interpreted the *letter as a specific unknown* (Kuchemann, 1981). Although 7% of learners wrote a correct step,  $4(n+5)$ , we cannot assume that they would be able to execute multiplication correctly. About 83% of learners responded incorrectly by giving responses such as  $n+20$ ,  $20n$ ,  $4n+5$ ,  $20$ ,  $24$ ,  $45$  and  $4n^5$ . Therefore, this question was one of the areas of focus for interviews with 2012 grade 9 learners.



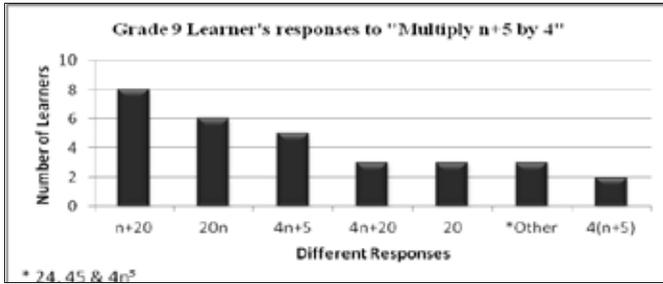


Figure 1: The Bar graph of different learners' responses

In another question learners were asked to "Evaluate  $e+f+g$  if  $e+f=8$ ". Of interest here was how learner would deal with the letter  $g$ . Approximately 17% of learners produced correct responses while a larger proportion 83% of learners produced incorrect responses. Examples of incorrect responses are drawn from Learner 8 and Learner 28 and I present them below.

If  $e+f=8$ ,  
 then  $e+f+g = \frac{8g}{8+g}$

Learner 8's response

If  $e+f=8$ ,  
 then  $e+f+g = 12$

Learner 28's response

The range of learners' responses is illustrated in figure 2. This question was followed up in the task-based interview.

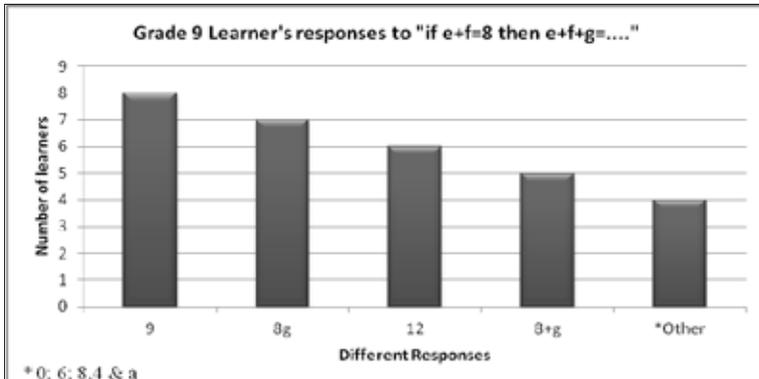


Figure 2: The Bar graph of learners' responses

### Task-based interview analysis

A summary of task-based interview analysis is presented in Table 1. Column 1 contains each of the tasks given to the 6 learners during the interview. Column 2 contains the frequencies of correct responses as well as their percentages. Column 3 consists of the frequencies of incorrect responses together with their percentages. Column 4 contains the instances of common errors that were present in learners' responses (for detailed learners' responses see Appendix 1).

<i>Task</i>	<i>Correct response</i>	<i>Incorrect response</i>	<i>Nature of Common errors as noted in learners' incorrect responses</i>
1. Add 5 to $3x$	2 (33%)	4 (67%)	<ul style="list-style-type: none"> <li>• Conjoined the terms</li> </ul>
2. Simplify: $3x + 2 + x$	2 (33%)	4 (67%)	<ul style="list-style-type: none"> <li>• Conjoined the terms</li> <li>• Interference of new knowledge</li> </ul>
3. Multiply $3x + 1$ by 5	3 (50%)	3 (50%)	<ul style="list-style-type: none"> <li>• Ignored brackets</li> <li>• Conjoined the terms</li> </ul>
4. Multiply $3x + 1$ by $x$	1 (17%)	5 (83%)	<ul style="list-style-type: none"> <li>• Ignored brackets</li> <li>• Conjoined the terms</li> <li>• Interference of new knowledge</li> </ul>
5. Simplify: $2(3x + 1)$	3 (50%)	3 (50%)	<ul style="list-style-type: none"> <li>• Conjoined the terms</li> <li>• Ignored brackets</li> </ul>
6. If $x + y = 10$ , then $x + y + z =$	1 (17%)	5 (83%)	<ul style="list-style-type: none"> <li>• Conjoined the terms</li> <li>• Substituted the letter with numeric value.</li> </ul>

Table 1: Summary of six learners' responses towards the interview task

### Task instruction contributed to learner errors

It appears that the task instructions in the interview contributed to some of the errors learners made. This is clearly demonstrated in question 1, where I asked learners to "Add 5 to  $3x$ ". In Booth's (1999) terms, they interpreted addition as an "action symbol" instead of seeing it as part of the solution, which gave rise to conjoining of terms. Booth (1999) advocates that learners tend to simplify an algebraic solution to a single term. The extract below indicates the learner's explanation concerning the instructions to the questions.



### Excerpt 1: Learner A and Learner B's explanations

The learners' response is given in the appendix.

- Researcher: Now I am looking at number 3, 4, 5 and looking at your answers and comparing with 1 & 2. In 1 & 2 you only have one answer but why here (pointing at 3, 4 & 5) you do not have one answer?
- Learner A: Because madam here (referring to number 1 & 2) they said add and here (referring to number 3, 4 & 5) they said I must multiply.
- Researcher: Oh! Because in number one you were adding and here, you are multiplying.
- Learner A: Yes, madam.
- Researcher: Why did you not add  $15x$  and  $5$ ? (referring to number 3)
- Learner A: Because they are not like-terms.
- Researcher: What about  $5$  and  $3x$  (referring to number 1)?
- Learner A: Because in number one they said add.
- Researcher: Oh! Because of the instruction.
- Learner A: Yes, madam I added  $5$  and  $3x$ .
- Learner B: In number one, they said add  $5$  to  $3x$ . Because they said **add**, me I also added but I know that they are not the like-terms. I understand that they said add that is why I added.

Excerpt 1 indicated that the learner's experience of working with numbers prompted them to treat the instruction as in numeric tasks for example, add  $5$  to  $3$ . Although these learners knew that  $5$  and  $3x$  were not like-terms but because of their experience with arithmetic context they conjoined the two terms. Again, the instruction 'simplify' in question 2 prompted learners to write their answer in a simplest form of a single term.

### Interference of new knowledge gave rise to errors

Learning new concepts in algebra appears to interfere with previous knowledge (Olivier, 1989; MacGregor & Stacey, 1997 & 1999). This was demonstrated by Learner A, when simplifying algebraic expression in question 2. Although Learner A mentioned grouping of like-terms, it is apparent that s/he treated the algebraic expression as an equation. See the extract below.

### Excerpt 2: Learner A's explanation

- Learner A: They said  $3x+2+x$  then I said e... when  $x$  comes between  $3x$  and  $2$  it changes the sign to negative  $x$ , then I said  $3x-x+2$ , then I got the answer for  $3x-x$  which is  $2x$  and I left the  $2$  there, then I said  $2x+2$  which gave me  $4x$ .
- Researcher: Let us go back to where you added the like-terms. You said when  $x$  moves closer to  $3x$  it changes the sign. Why?
- Learner A: Because that is how my teacher taught me that when  $x$  e... when the equation moves to the other side it changes the sign.
- Researcher: Oh! When it is an equation. Is this an equation?
- Learner A: Yes, madam.
- Researcher: Why do you say it is an equation?
- Learner A: Because madam it has the variables.

Similarly, Learner E demonstrated when multiplying  $3x+1$  by  $x$  that the knowledge of exponents interfered which gave rise to errors. Below I provide the extract of the learner's explanation.

### Excerpt 3: Learner E's explanation

- Learner E: They said we must multiply  $3x+1$  by  $x$ , so I did the same thing as in number 3 I put  $3x+1$  in brackets and  $x$  outside the brackets then multiplied  $3x$  by  $x$  which is  $3x^2$  and  $x$  times  $1$  which is  $x$  then  $x$  has the power of one. I said  $3x$  and added the exponent  $2$  plus  $1$  which gave me  $3x^3$ .
- Researcher: (the interviewer did not follow the  $3x^3$  since it was outside the scope of the interview).

In excerpt 2, it is clear that the learner could not differentiate between an expression and an equation. This could be the result of the new knowledge of equations which is not deeply rooted (Olivier, 1989). Again, although the learner in excerpt 3 was doing well with other tasks, the errors she produced may have been a result of applying her new knowledge of exponential laws. This shows the diversity of errors produced in introductory algebra. Therefore, the interference of new knowledge portrayed the complexity of working with algebraic problems.

### Ignoring a letter gave rise to errors

It was clear that some learners isolate the letter as they continue with some aspects of the procedure. In Kuchemann's (1981) terms this is considered as ignoring the letter. This is clearly demonstrated by learner C who said, "5 plus 3 equal to 8 then I took the  $x$  and put it next to 8 to get  $8x$ ". This learner had definitely ignored the letter while doing calculations and attached it in the final response. Below I present the extract of Learner C for question 1, 2 and 3.



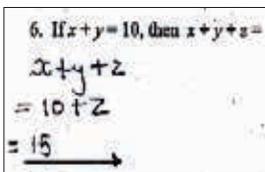
#### Excerpt 4: Learner C's explanation

- Learner C: They said add 5 to  $3x$  so I said 5 plus 3 equal to 8 then after that I took the  $x$  and put it next to 8 to get  $8x$ .
- Learner C: They said simplify, is it that there is  $3x+2+x$ , so I said 3 plus 2 equal to 5 then I took the  $x$  and put it.
- Learner C: They say multiply  $3x + 1$  by  $x$ , so here I did the same thing I took 3 and 1 and added together and got 4 then I took the  $x$  put it next to 4 and got  $4x$ .

In excerpt 4, it is clear that the learner did not recognize the letter  $x$  in question 2 and 3. Hence, s/he chose to operate with numeric values. Again learners' experience with numeric context hinders them from dealing with algebraic context. Therefore, this learner reasoned from the numeric perspective which prompted him/her to view the letter as an object.

#### Replacing a letter with numeric value gave rise to errors

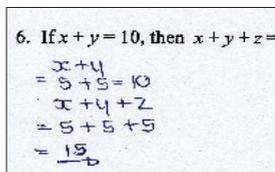
Learners tend to assign a numeric value for letters when simplifying algebraic expressions. According to Christou *et al.*, the tendency of learners "to use their prior experience with numbers in the context of arithmetic" (2007, p. 289) interferes with the interpretation of letters in algebra. This was demonstrated by Learner D and E when they used 5 to replace  $x$ ,  $y$  and  $z$  (see their response below).



6. If  $x+y=10$ , then  $x+y+z=$

$$\begin{aligned} x+y+z \\ = 10+z \\ = 15 \end{aligned}$$

Learner D's response



6. If  $x+y=10$ , then  $x+y+z=$

$$\begin{aligned} x+y \\ = 5+5=10 \\ x+y+z \\ = 5+5+5 \\ = 15 \end{aligned}$$

Learner E's response

These two learners had similar explanations. Below I provide the extract of Learner D's explanation for the response.

#### Excerpt 5: Learner D's explanation

- Learner D: So by number six they say if  $x+y=10$  then  $x+y+z$  is equal to what? So I said  $x+y+z = 10 + z$  which is the ten I got from  $x+y$  then equal to 15, because each number is five because if you say  $5+5=10$  so I thought maybe they said  $5+5$  so I gave the  $z$  also a five which gave me 15.
- Researcher: If you say,  $5+5$  is equal to 10. Are those the only numbers that can give us 10?
- Learner D: No madam
- Researcher: Ok, give me a set of two numbers that add up to ten.
- Learner D: Eight plus two.
- Researcher: Why did you not use eight and two?
- Learner D: I gave each the equal amount of number.

Excerpt 5 indicated that the learner could justify his/her strategy to assign equal value for  $x$ ,  $y$  and  $z$ . These learners evaluated the letter by substituting but didn't operate on the letters as unknowns.

## CONCLUSION

The research seeks to explore learner thinking underlying explanations of their errors in introductory algebra. The findings indicated that the diversity of errors produced by learners are influenced by to task instruction, new knowledge, ignoring the letter and replacing letter with numeric value as well as other factors. This clearly indicated the complexity of dealing with algebraic expressions. It was noted from the excerpts that learners who seemed to understand like-terms produced errors as a result of task instructions and interference of new knowledge.

Listening to these learners' explanation made me realize that learners grasp information in different ways and respond according to what they have learnt. For example, the learner that said an expression is an equation because it has variables is partly correct but needs to extend that knowledge to observe that an equal sign plays an important role in his definition. This suggests that many errors result from knowledge that is used inappropriately such as over-generalising from instances where it may be appropriate to instances where it does not apply. Moreover, this research project has influenced my practice as a mathematics teacher to be more aware of errors, mistakes and misconceptions that are likely to occur in algebra. I have learnt to accept and appreciate these errors in my classroom and constantly address them to alert learners.

I would conclude by giving teachers a word of advice that these errors do happen in our classroom, however the extent in which one deals with them will determine the learners' conceptual understanding in algebra.

## ACKNOWLEDGEMENT

I would like to acknowledge Wits Maths Connect Secondary project for granting me permission to use its data in this paper. Being part of the Transition Maths 1 course illuminated so many things that were happening in my classroom which I could not deal with. Again, thank you WMCS for providing teachers opportunities to participate in professional development. I would also like to thank Shadrack Moalosi from WMCS for his help in writing this paper.



**APPENDIX 1: 2012 GRADE 9 RESPONSES ON THE TASK-BASED INTERVIEW**

<i>Learners</i>	<i>Learners' responses to the tasks</i>					
	<i>1</i> <i>Add 5 to 3x</i>	<i>2</i> <i>Simplify:</i> $3x + 2 + x$	<i>3</i> <i>Multiply</i> $3x + 1$ <i>by 5</i>	<i>4</i> <i>Multiply</i> $3x + 1$ <i>by x</i>	<i>5</i> <i>Simplify:</i> $2(3x + 1)$	<i>6</i> <i>If <math>x + y = 10</math>, then <math>x + y + z =</math></i>
<i>A</i>	$5+3x = 8x$	$3x-x+2 = 2x+2 = 4x$	$5(3x+1) = 15x+5$	$x(3x+1) = 3x^2+x$	$= 6x+2$	$= 10+z = 10z$
<i>B</i>	$5+3x = 8x$	$3x+3x = 6x$	$3x \times 6 = 18x$	$3x+2x = 5x$	$6x+2 = 8x$	$= 10$
<i>C</i>	$8x$	$5x$	$= 4x \times 5 = 20x$	$= 4x \times x = 4x^2$	$2 \times 3x+1 = 7x$	$10z$
<i>D</i>	$5+3x = 8x$	$= 5x+x = 6x$	$3x+1 \times 5 = 4x \times 5 = 20x$	$3x+1 \times x = 4x \times x = 4x^2$	$6x+2 = 8x$	$10+z = 15$
<i>E</i>	$5+3x = 3x+5$	$= 3x+x+2 = 4x+2$	$5(3x+1) = 15x+5$	$x(3x+1) = 3x^2+x = 3x^{2+1} = 3x^3$	$= 6x+2$	$x+y=5+5=10$ $x+y+z=5+5+5 = 15$
<i>F</i>	$3x+5$	$3x+x+2 = 4x+2$	$5(3x+1) = 15x+5$	$x(3x+1) = 3x+x = 4x$	$6x+2$	$10+z$

**REFERENCES**

Booth, L. (1999). *Children's difficulties in beginning algebra*. In B. Moses (Ed.), *Algebraic Thinking, Grades K-12* (pp. 299–307). Reston, VA: NCTM.

Christou, K.P., Vosniadou, S. & Vamvakoussi, X. (2007). *Students' interpretation of literal symbols in algebra*, Chapter 19, 285-299. *Reframing the Conceptual Change Approach in Learning and Instructions*: Elsevier Ltd.

Creswell, J.W. (1998). *Qualitative inquiry and research design: Choosing among five traditions*. Thousand Oaks London: SAGE Publishers, International Educational & Professional Publisher.

Department of Basic Education (DBE), (2011). *Report on the qualitative analysis of ANA results*, Department of Education: South Africa, Pretoria.

Department of Education (DoE), (2002). *Revised National Curriculum Statement for Mathematics*, Department of Education: South Africa, Pretoria.

Kuchemann, D. (1981). *Algebra in K. Hart (ed.). Children's understanding of mathematics*, 11-16. London: John Murray, pp 102-119.

MacGregor, S. & Stacey, K. (1997). *Students' understanding of algebraic notation*. *Educational Studies in Mathematics*, 33, 1-19. Kluwer Academic Publishers.

- Olivier, A. (1989). *Handling pupils' misconceptions*. Presidential address delivered at the Thirteenth National Convention on Mathematics, Physics and Biology Education, Pretoria, 3-7 July.
- Simkins, C. (2013). Performance in the South African educational system: What do we know? Johannesburg: CDE.
- Stacey, K. & MacGregor, M. (1999). Ideas about symbolism that students bring to algebra. In B. Moses (Ed.), *Algebraic Thinking Grades K-12* (pp. 308-312). Reston, VA: NCTM.
- Stacey, K., & MacGregor, M. (1994). *Algebraic Sums and Products: Students' Concepts and Symbolism*. In da Ponte, J.P. & Matos, J.F. (eds.) *Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education (PME XVIII) (Vol.III)*: Department of Education, University of Lisbon, Portugal, 289-296.
- Vermeulen, N. (2007). *Does Curriculum 2005 promote successful learning of elementary algebra?* *Pythagoras* 66, 14-33.



## DEEPENING THINKING-LIKE PROBLEMS: THE CASE OF TWO STUDENTS

Bruce May and Cyril Julie

University of the Western Cape

*Pilot findings of a teaching model based on spiral revision, productive practice and a together-and-apart (TAP) strategy is presented. Subsequently the constituent parts of the teaching model are discussed. A deepening-thinking-like example is used to provide insight on implementation of the model. The conceptual framework based on mathematical reasoning is presented and linked to the study. A classroom instance emerging from implementation of the model is narrated. The classroom instance is analyzed and a possible conclusion is offered.*

### INTRODUCTION

The term often used by the print media in South Africa to describe the state of mathematics education in the country is crisis. Large scale research such as the Trends in International Mathematics and Science Study (TIMSS) (Mullins, Martin, Foy & Arora, 2012) and the Annual National Assessments (ANA's) (DBE, 2012) paints a bleak picture of the mathematical proficiency of South African learners. Also more often than not mathematics teachers have been identified by print media authors as the main protagonists in this 'crisis'. Many of these authors maintain that poor subject matter knowledge of teachers contribute to the dismal performance of students in mathematics. Now although these criticisms are unfair generalizations since they seem to include all South African teachers irrespective of their success in teaching mathematics, it cannot be denied that South Africa has major problems in the teaching and learning of mathematics. A relevant question therefore is what strategies can be used to enhance the mathematical knowledge and cognitive abilities of pre-service teachers?

Shulman (1986) distinguishes between three categories of teacher content knowledge namely subject matter content knowledge, pedagogical content knowledge and curricular content knowledge. Although the organization, composition and characteristics of mathematical content knowledge for teaching has been extensively researched there is no consensus among researchers concerning the mathematics teachers need to know in order to deliver effective teaching (Ball, Hill & Schilling, 2004). Furthermore although research has shown that teachers' mathematical knowledge is significantly related to learner achievement, the nature and extent of that knowledge is not known (Ball, Hill & Rowan, 2005).

We contend it is the complexity involved in the study of the teaching and learning of mathematics that limits us to research narrow areas of mathematics education, since if we attempt to focus research on too many issues at once we will not be able to sufficiently interrogate each of the issues adequately to provide meaningful contributions to the field. Hence the research described in this paper will focus on how a specific adapted teaching approach influences pre-service teachers in terms of presence and range of competencies in relation to specific mathematical activities (Niss, 2003). The aim of the study therefore is to investigate if a teaching model based on spiral revision, deepening thinking-like problems and TAP (Together-and-Apart) strategies do or do not enhance the procedural and conceptual knowledge and ways of mathematical reasoning of pre-service mathematics teachers. The overarching philosophy of the teaching model is that the teacher should become less and the student more during their regular engagements. That is students should do more and communicate more during lessons and the teacher less. The research will not focus on the teaching model, but rather on an issue emerging from the implementation of the teaching model. It will focus on one classroom instance and zero in on the work of two pre-service mathematics teachers. This is part of a larger project investigating meaningful mathematical education for pre-service teachers being trained to teach at the General Education and Training (GET) level. The teaching model, theoretical framework, conceptual underpinnings of the study and a classroom instance will be discussed in the sections that follow.

### THE TEACHING MODEL

One of the cornerstones of the model is spiral (or repeated revision) revision which is defined as the recurrent practising of previously covered mathematical work in specified content areas (Julie, 2013). This is linked to the notion of working memory. When students are required to solve mathematical tasks using newly learned procedures or concepts their working memories can easily be overloaded since they must deal with many new elements of information at once. Conversely, if students had some practice in a mathematical content area they can use their existing knowledge structures to make inferences and make connections between well entrenched concepts in their long term memories to solve mathematical problems presented to them (Cronbach & Snow, 1977; Kalyuga, 2007; Durkin, Rittle-Johnson & Star, 2009). The importance of practicing procedural skills to the extent that it becomes part of the long term memory (automatization) cannot be underestimated. A reason that we advance for the need for spiral revision is that students sometimes struggle with mathematics because they have not practised lower level procedural skills to the extent that it became part of their long-term memories.

What we are advocating is that completed work be revised in class on an on-going basis right through the semester. The idea is that tasks that are conceptually and procedurally more demanding are presented to students during each subsequent cycle of the revision process.



Although the majority of revision problems would be restricted to a specified content area or concept (i.e. the problems would require knowledge from only one content area or concept) some problems presented to students would require integrated knowledge. A major problem in the learning of mathematics is that students tend to compartmentalize mathematical knowledge. Ball (1988) argues that mathematics in the school curriculum is presented in compartments and mathematical content is treated as a collection of discrete bits of procedural knowledge. A consequence of this tendency to compartmentalize mathematical knowledge is that the cognitive load required in knowing and using mathematics is considerably increased. Students instructed in this manner will have a low level of knowledge integration and as a result knowledge accessibility will be negatively affected. Consequently the revision that we envisage should also include tasks whose solutions are based on a combination of two or more concepts from different content areas that have been covered previously and should where possible also require the use of flexible procedural knowledge.

Since it is our intention to use spiral revision to enhance the procedural and conceptual knowledge of pre-service teachers we next discuss these essential knowledge components. It is generally accepted in the mathematics community that conceptual and procedural knowledge are essential knowledge components in the learning of mathematics. Hiebert and Lefevre (1986) contend that conceptual knowledge is knowledge that is rich in relationships. This connected web of knowledge is a network in which the linking relationships are as prominent as the discrete pieces of information. Procedural knowledge is knowledge that consists of rules and procedures for solving mathematical problems. Procedural knowledge also consists of knowledge of mathematical symbols and the syntactic conventions for the manipulation of such symbols (Hiebert & Lefevre, 1986; Star, 2005). Star (2005) argues that skilled problem solvers in mathematics are also flexible in their use of known procedures. A student that does not possess such flexible procedural knowledge sometimes will not be able to solve unfamiliar problems where the solution requires the student to use known procedural knowledge. The student will also not be able to produce a maximally efficient solution in the absence of such flexible procedural knowledge. A result therefore of such flexibility is that students that possess such knowledge will have the ability to generate maximally efficient solutions for known and even sometimes unknown problem situations. Star (2005) contends that flexible procedural knowledge is deep procedural knowledge and would allow a student that possess such knowledge to use mathematical procedures that would best fit a provided known or novel problem situation.

Hiebert and Lefevre (1986) argue that sound mathematical knowledge includes significant and fundamental cognitive links between procedural and conceptual knowledge. They maintain that for students to be competent in mathematics they need to possess both procedural and conceptual knowledge and cognitive links between these two essential knowledge components. They describe rote learning as learning that produces knowledge that do not include relationships with other knowledge and which is closely tied to the context in which it was learned. The consequence is that knowledge acquired through rote learning can only be accessed and applied in contexts that mirror the context in which it was learned.

In the model spiral revision is used in conjunction with productive practice. Julie (2013) argues that productive practice is a strategy where students are exposed to deepening thinking-like problems. Deepening thinking-like problems are utilized to enrich the conceptual knowledge of students in requisite content areas of the specified mathematics curriculum.

Students entering initial professional teacher education programmes come from diverse backgrounds, with varying proficiency levels in the various disciplines. This diversity is very prominent in mathematics and consequently teaching such students require a teaching approach that incorporates methods that deal with diversity. The Together- and -Apart (TAP) teaching approach is an attempt to deal with such diversity. The goal of TAP according to its authors (Bennie et al, 2000) is to achieve equity in the mathematics classroom by acknowledging learner diversity in some teaching instances and ignoring diversity in other teaching instances.

### **A DEEPENING THINKING-LIKE EXAMPLE**

Problem solving in mathematics require students to view mathematical concepts from different angles. For example a function can be viewed in terms of its operational character, as a process of co-variation (i.e. how the dependent variable co-varies with the independent variable) or as a mathematical object (Boon, Doorman, Drijvers, Gravemeijer & Reed, 2012). An example of such questions is:

- For what values of  $x$  will:  $\{(2x; 2x - 1); (x^2 - 3; 3x), x \in \mathbb{R}\}$  not be a function?

This question requires the student to think of the function in terms of co-variation, but importantly also to utilize the definition of the function concept to solve the problem and in so doing perhaps enhancing understanding of the function definition. The solution strategy is also based on the exploitation of a known procedure. The student however has to make the connection between the concept and the procedure. This is our objective with deepening thinking-like questions i.e. to enhance and to deepen conceptual knowledge of students and for the student to make cognitive connections between procedural and conceptual knowledge.



The following solution elucidates the above claims:

Functions can be defined in more than one way. For our purposes we will use the following definition:

*A function  $f$  from a set  $A$  to a set  $B$  is a relation that assigns to each element  $x$  in the set  $A$  exactly one element  $y$  in the set  $B$ . The set  $A$  is the domain of the function and the set  $B$  is the range of  $f$ .*

This definition implies the following:

1. *Each element in the domain must be matched with an element in the range.*
2. *Some elements in the range may not be matched with any element in the domain.*
3. *Two or more elements in the domain may be matched with the same element in the range.*
4. *An element in the domain cannot be matched with two different elements in the range.*

Since the question requires us to find those  $x$  values that will cause the ordered pairs not to represent a function we utilize statement 4 above. That is we want the first coordinates to be the same and the second coordinates different. What we therefore are attempting to achieve is to make explicit the fact that a first coordinate cannot be matched with two different second coordinates. The student would therefore be forced to really think about what the definition implies. The following solution explicates this. We start by equating the first coordinates i.e.

$$x^2 - 3 = 2x$$

One should then realize that a known procedure could be utilized to solve for  $x$  i.e. one could use solution of quadratic equations. That is the student should make a connection between the procedure and the concept.

$$\begin{aligned}x^2 - 2x - 3 &= 0 \\(x + 1)(x - 3) &= 0\end{aligned}$$

Which provide two possible solutions i.e.

$$x = -1 \text{ or } x = 3$$

We have to check which  $x$ -value gives the desired result by substituting into the original coordinate pairs i.e.

$x = -1$ , yields  $\{(-2, -3); (-2, -3)\}$ . This is not the desired result.

If  $x = 3$ , then we have  $\{(6,5); (6,9)\}$ . Which is the desired result, since we now have the first coordinates the same and the second coordinates different which violates the 4<sup>th</sup> statement above i.e. that an element in the domain cannot be matched with two different elements in the range.

## IMITATIVE AND CREATIVE REASONING

The conceptual framework underpinning the investigation is based on mathematical reasoning. Lithner (2008) has developed a conceptual framework that is concerned with reasoning in mathematics. In this framework reasoning is defined as the line of thought required to produce assertions and reach conclusions in solving mathematical tasks. Reasoning in this framework can either be the thinking processes or the product of thinking processes or both. A line of thought may be classified as reasoning even if it is incorrect. The only proviso is that it makes sense to the thinker.

Lithner (2008) differentiates between two different types of reasoning in mathematics. Imitative reasoning (IR) occurs when a student produces a solution procedure that he/she memorized. Conversely creative reasoning (CR) is reasoning that is characterized by flexibility and novel approaches to mathematical problems (Bergqvist, 2007).

Lithner (2008) distinguishes between two main two main categories of imitative reasoning, namely memorized and algorithmic imitative (AR) reasoning. He asserts that for imitative reasoning to be classified as memorized reasoning it needs to fulfil two conditions. On the one hand the reasoning should be based on recalling a complete answer. On the other hand the implementation strategy should consist of only writing the answer down. A reasoning sequence is classified as algorithmic reasoning (AR) if the reasoning is based on the recall of an algorithm. An algorithm is described as a finite sequence of executable directives which allows one to find a definite result for a given class of problems. A reasoning sequence is classified as algorithmic reasoning if it satisfies two conditions. On the one hand the strategy choice for the reasoning should be to recall an algorithm as a solution. No other reasoning should be required except to implement the algorithm. Bergqvist (2007) claims that in some instances students use algorithmic reasoning to solve mathematical tasks, without any comprehensive understanding of the underlying mathematical concepts.

Bergqvist (2007) describes creative reasoning (CR) as reasoning that is not hindered by fixation and is characterized by flexibility, novel approaches to mathematical problems and well-founded task solutions. If a task is very nearly solvable using imitative reasoning and creative reasoning is only required to modify an algorithm the reasoning required is local creative reasoning (LCR). On the other hand if a task requires mostly creative reasoning then the reasoning involved is classified as global creative reasoning (GCR).



We are of the opinion that in the majority of cases in South Africa the learning process in mathematics is started with imitative reasoning. In this process the learners imitate the written steps of the procedures of solutions to mathematical tasks that they are shown by their instructor. The idea is that the students not only internalize the external manifestations of the thought processes of the teacher (i.e. the written steps or algorithms) but that they also are aware of and understand the ideas that connect the individual steps and the solution process as a whole. Note I am not contending that students internalize ideas from teachers as a mechanical transaction, but that the process of internalization is underpinned by meaning making by the student. We however ideally want our students to move beyond using only imitative reasoning to instances where they use prior knowledge creatively. The question is what pedagogy would aid in such an endeavour?

### A CLASSROOM INSTANCE

The following is a narrative of a classroom incident that occurred when the teaching model was piloted by the lead author. The topic under discussion in this lesson was the notion of rationalisation of denominators. In the lesson rationalisation of denominators of examples such as  $\frac{12}{\sqrt{18}}$ ;  $\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$  and  $\frac{1}{\sqrt[3]{6}}$  were discussed. After solutions were provided and discussed for these and other similar examples student A posed a verbal question. Since the lead author did not quite understand his question the student was requested to write his question on the white board. The following was his question. "How would one do this one  $\frac{a}{\sqrt[n]{b}}$ ". The lead author was in the process of providing a longwinded explanation when student A interjected and asked if he could supply a solution. The following is his solution:

$$\begin{aligned} & \frac{a}{\sqrt[n]{b}} \times \frac{\sqrt[n]{b^{n-1}}}{\sqrt[n]{b^{n-1}}} \\ &= \frac{a \cdot \sqrt[n]{b^{n-1}}}{\sqrt[n]{b^{n-1} \cdot b}} \end{aligned}$$

In order to rationalize denominators the notion of one in a different form is utilized as well as the fact that one is the identity element for multiplication in the real number system. The provided solution suggests that student A is aware of both of these ideas. The fact that he did not continue the solution beyond this second step perhaps also suggests that he is aware of the fact that after multiplication is effected the exponent under the root should be equal to the order of the root. His question seems to suggest also that he was wondering what would happen in the case of a variable root. Since all the examples discussed in the lesson utilized integer roots his question extends the discussion into a new direction i.e. what would happen in the case of a general root.

It would seem therefore that student A was attempting to widen or extend the domain of this type of task. Mitchelmore (2002) classifies such attempts as empirical extension generalisation. Such a question can be classified as a higher cognitive question since its solution requires local creative reasoning based on flexible procedural knowledge. Student A does not specify any restrictions on the variables  $a, b, n$  and neither does he specifies what kind of numbers  $a, b$  and  $n$  is.

In engagements between students and teacher in the mathematics classroom it is normally the mathematics teacher that provides questions in order to stimulate students' critical thinking. In the above case the converse happened i.e. student A provided a question that stimulated higher order thinking. Since I was curious as to his level of proficiency with this type of problem I presented student A and the class with the following deepening thinking-like follow-up question:

Simplify the following by rationalising the denominator:  $\frac{1}{n\sqrt{\frac{-c}{am^2-m-2}}}$

It should be noted that in the part under the root  $\frac{-c}{m^2-m-2}$  is the exponent of  $a$ .

The intention on the one hand was to revise the work covered previously and on the other to determine if student A would recognise that his previous solution strategy need only be modified slightly to solve this question and therefore there is some generalizability in the solution strategy. Student A however was stumped by this question and could not even start. Student B whom I thought to be primarily an imitative reasoner then claimed that he knew how to solve the problem. The following is his solution:

$$\begin{aligned} & \frac{1}{n\sqrt{\frac{-c}{am^2-m-2}}} \times \frac{n\sqrt{\frac{c}{am^2-m-2}+n}}{n\sqrt{\frac{c}{am^2-m-2}+n}} \\ &= \frac{n\sqrt{\frac{c}{am^2-m-2}+n}}{n\sqrt{a^n}} \end{aligned}$$

When I asked him how he arrived at his answer the following dialogue ensued.

Lead author: Why did you decide to multiply by  $\frac{n\sqrt{\frac{c}{am^2-m-2}+n}}{n\sqrt{\frac{c}{am^2-m-2}+n}}$  ?



Student B: I know that I have to take the additive inverse of the exponent since this is what we have learnt in the first semester when we were doing properties of rational numbers.

Lead author: Why did you add  $n$ ?

Student B: So that I can end up with  $a$  to the power  $n$ .

Haylock (1997) asks the question why a person that knows all the mathematics they need to solve a particular problem still fails to solve it. He contends that some reasons for this might be that the persons' mind is set in an inappropriate direction or that the person is adhering rigidly to an approach that does not lead to a solution. Another possibility might be that the problem solver simply does not make a cognitive connection between known prior requisite knowledge and the provided problem.

I think what we ideally want in the teaching and learning of mathematics is that students develop the ability to use prior knowledge not only flexibly, but also to recognise instances and new contexts where prior knowledge can be applied. Student A could not make the connection between his previous strategy and this new problem although in the previous lesson he was applying prior knowledge in a new way to ask a question that required an extension and modification of the reasoning requirements for such tasks. That is student A was utilizing creative reasoning in the previous lesson, but could not sustain this way of reasoning in the subsequent lesson. Student B however who did not in any of our previous engagements exhibit creative reasoning was able to not only make connections between concepts of the previous lesson, but also with concepts of lessons of the previous semester.

## CONCLUSION

The question is what if any conclusions can one draw from the above discussion. Perhaps the above described interactions of students A and B is an indication that if we allow and encourage students from diverse backgrounds to grapple with deepening thinking-like problems then the possibility opens up for them to display their creativity and move beyond concrete primarily number-driven examples to more generalised formulations of some mathematical construct. We are however of the opinion that even with the utilization of a teaching model it takes a long time to change significantly the knowledge levels and cognitive abilities of students. However if a student is willing to attempt and to continue to attempt then the student would as a result of learnings gleaned from these attempts enhance their cognitive abilities and knowledge levels in mathematics. Further investigation is required to determine if the above described teaching model augments the willingness of students to attempt.

## REFERENCES

- Ball, D. (1988). The subject matter preparation of prospective mathematics teachers: Challenging the myths. Researcher Report 88-3. East Lansing, MI: Michigan State University, National Center for Research and Teaching.
- Ball, D., Hill, H.C., Schilling, S.G. (2004). Developing Measures of Teachers' Mathematics Knowledge for Teaching. *The Elementary School Journal*, 105(1), 11 – 30.
- Ball, D.L., Hill, H.C., Rowan, B. (2005). Effects of Teachers' Mathematical Knowledge for Teaching on Student Achievement. *American Educational Research Journal*, 42(2), 371 – 406.
- Bergqvist, E. (2007). Types of reasoning required in university exams in mathematics. *Journal of Mathematical Behavior*, 26, 348 – 370.
- Cronbach, L.J. & Snow, R.E. (1977). *Aptitudes and Instructional Methods*. New York: Irvington.
- DBE (Department of Basic Education) (2012). Report on the Annual National Assessment of 2012. Pretoria: DBE.
- Doorman, M., Drijvers, K., Gravemeijer, P., Boon, P. & Reed, H. (2012). Tool use and the development of the function concept: From repeated calculations to functional thinking. *International Journal of Science and Mathematics Education*, 10, 1243 – 1267.
- Durkin, K., Rittle-Johnson, B., Star, J.R. (2009). The Importance of Prior Knowledge When Comparing Examples: Influences on Conceptual and Procedural Knowledge of Equation Solving. *Journal of Educational Psychology*, 101(4), 836 – 852.
- Haylock, D. (1997). Recognizing mathematical creativity in schoolchildren. *Zentralblatt Fuer Didaktik der Mathematik*, 29(3), 68 – 74.
- Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1 – 27). Mahwah, NJ: Lawrence Erlbaum Associates.
- Kalyuga, S. (2007). Expertise Reversal Effect and Its Implications for Learner- Tailored Instruction. *Educational Psychology Review*, 19: 509 – 539.
- Linchevski, L., Kutscher, B., Olivier, A., & Bennie, K. (2000). Indispensable Mathematical Knowledge - IMK and Differential Mathematical Knowledge – DMK: Two Sides of the Equity Coin. In T. Nakahara & M. Koyama (Eds.), *Proceedings of the Twenty fourth International Conference for the Psychology of Mathematics Education: Vol. 3*, (pp. 239 – 246). Hiroshima, Japan.
- Lithner, J. (2008). A research framework for creative and imitative reasoning. *Educational Studies in Mathematics*, 67(3), 255 – 276.
- Julie, C. (2013). Towards a model for intentional teaching for improving achievement in high-stakes mathematics examinations. In Z. Davis & S. Jaffer (Eds.), *Proceedings of the 19<sup>th</sup> Annual Congress of the Association for Mathematics Education of South Africa, Vol. 1*. (pp. 86 – 96). Cape Town: AMESA.
- Mitchelmore, M. (2002). The role of abstraction and generalisation in the development of mathematical knowledge. *Proceedings of the East Asia Regional Conference on Mathematics Education (EARCOME)*, 1, 157 – 167
- Mullins, I.V.S., Martin, M.O., Foy, P and Arora, A. (2012). *TIMSS 2011 International Results in Mathematics*. Chestnut Hill, MA: Boston College.
- Niss, M. (2003). Mathematical competencies and the learning of mathematics: The Danish KOM project. In Gagatsis, A., & Papastavridis, S. (Eds), *3<sup>rd</sup> Mediterranean Conference on Mathematical Education*, (pp. 115 – 124). Athens, Greece: Hellenic Mathematical Society and Cyprus Mathematical Society.
- Shulman, S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4 – 14.
- Star. J.R. (2005). Reconceptualising Procedural knowledge. *Journal for Research in Mathematics Education*.



## A MATHEMATICAL PROBLEM-SOLVING EXERCISE BY IN-SERVICE TEACHERS IN A CONTINUOUS TEACHING DEVELOPMENT CONTEXT.

Monde Mbekwa and Rajendran Govender.  
University of the Western Cape

*This paper is a reflective description of a problem-solving exercise by 40 in-service teachers in a mathematics continuous development workshop. The exercise was aimed at assessing the teachers' subject content knowledge and problem-solving strategies when given a non-routine trigonometric identity to prove. The teachers were divided into three groups and given a time of approximately one hour to prove the identity. At the end of the allotted time, the teachers presented their solution. The solution reveals that teachers were able to solve the problem after following the problem-solving steps proposed by Polya and Schoenfeld's criteria, whilst one group did not complete the process.*

### INTRODUCTION

This paper reflects on a problem-solving exercise by in-service teachers participating in a mathematics continuous professional development project, the Local Evidence Driven Improvement in Mathematics Teaching and Learning Initiative (LEDIMTALI) led by a higher education institution in the Western Cape Province of South Africa. In this project mathematics teacher educators, mathematicians, mathematics teachers and mathematics curriculum advisors work collectively and collaboratively to facilitate quality teaching of mathematics. The initiative is premised on the belief that such collective and collaborative work can cascade to learners achieving at their highest potential in mathematics.

The LEDIMTALI project which started in earnest in 2012 endeavours to establish and develop a community of goal driven mathematics practitioners by:

- (a) Providing opportunities for reflections on classroom-based teaching of mathematics.
- (b) Designing and developing strategies to enhance ways of teaching mathematics based on results forthcoming from these considered reflections.
- (c) Supporting teaching of mathematics through coaching, training and co-teaching.
- (d) Providing resources to enhance both the teaching and learning of mathematics.
- (e) Developing opportunities for the enhancement and understanding of the appropriate mathematical knowledge that underlies the teaching of school mathematics.

- (f) Developing respectful ways of working amongst and between mathematics educators, mathematics, mathematics teachers and mathematics curriculum advisors for the enhancement of teaching mathematics. (Julie, 2011, p.1).

To achieve the goals articulated by the LEDIMTALI document, regular three hour workshops and are held and attended by all the stakeholders.

In addition bi-monthly teacher institutes are also organised as residential workshops whereby all participants stay over for a weekend for intensive discussion of the mathematics curriculum, pedagogy and general mathematical engagements.

Activities which generally take place at these institutes include the following:

- Reports and reflections on teaching experiments, workshops and school teaching experiences
- The teaching of mathematical concepts and design of lessons.
- Setting of end-of-year examination for grades 10 and 11
- Engaging with mathematical content on request by teachers or from proposal by teacher educators.

It is in the context of “Developing opportunities for the enhancement and understanding of the appropriate mathematical knowledge that underlies the teaching of school mathematics” (Julie, 2011) that one of the institutes dedicated a two hour session on problem-solving on the 18<sup>th</sup> October 2013 in . One of the problems in this workshop involved the proof of a trigonometric identity. This paper analyses and describes the strategies which teachers employed in proving the identity.

## LITERATURE REVIEW

The basic assumption for these kinds of mathematical engagement sessions in LEDIMTALI is the imperative of the enhancement of teacher knowledge of mathematical content on the understanding that this would have an effect on the teachers’ ability to teach the subject to their learners (Jadama, 2014). LEDIMTALI assumes that as part of continuous professional development and in addition to the discussion on curriculum issues, regular mathematical engagement should occur so that teachers’ knowledge should be constantly sharpened. As Shulman (1987) puts it “... Thus teaching necessarily begins with a teacher’s understanding of what is to be learned and how it is to be taught.” (p7). This is informed by the notion of teachers’ subject content knowledge which has received a lot of attention from mathematics educators recently. Much of the writing on this issue derive from the seminal work of Shulman (1987) who propounded a taxonomy of what he proposes should be a knowledge base of what teachers should possess. He proposes seven categories of teacher knowledge inter alia, content knowledge, pedagogical content knowledge and curriculum knowledge. He refers to subject content knowledge as “- the knowledge, understanding and skill, and disposition that are to be learned by school children.” (Shulman, 1987. p8).



In addition Shulman posits that the teacher should know more than the basic content namely how the subject matter is structured and organised and the kinds of questions that are pursued in the particular field of study. This is important because the teacher is primarily responsible for ensuring that students understand the subject matter. Most authors concur on the importance of the teachers' content knowledge as an important prerequisite for effective teaching (Fennema & Franke, 1992; Lucus, 2006; Ma, 1999; Norman, 1992; Stein, Baxter & Leinhardt, 1990)

This paper reflects on an exercise which gave teachers an opportunity to demonstrate their knowledge of trigonometry by participating in a trigonometric proof. This we regard as part of a process of problem-solving.

Problem-solving has been proposed in the last four or more decades as an objective for mathematics teaching and learning by many mathematical associations throughout the world. The American National Council of Teachers of Mathematics for instance in their *Principles and Standards for School Mathematics* (NCTM, 2000) state that problem-solving should not only be a goal for learning mathematics but should also be a means for doing so. The NCTM states that:

Problem-solving means engaging in a task for which the solution method is not known in advance. In order to find a solution, students must draw on their knowledge, and through this process, they will develop new mathematical understanding. ...Student should have frequent opportunities to formulate, grapple with and solve complex problems that require significant amount of effort and should then be encouraged to reflect on their thinking (NCTM, 2000, 52).

In South Africa, the new Curriculum and Assessment Policy Statement (CAPS) of the national Department of Basic Education (DBE) states that:

Problem-solving and cognitive development should be central to all mathematics teaching. Learning procedures and proofs without a good understanding of why they are important will leave learners ill-equipped to use their knowledge in later life. (p.8)

Many mathematics educators that we have engaged with propose that problem-solving should not only be seen as an activity that occurs in the classroom but should extend to the application of mathematics to real life outside the classroom. We believe that teachers themselves should experience problem-solving in the same ways that learners should.

Problem-solving can be regarded as:

...the means by which individuals take the skills and understandings they have developed previously and apply them to unfamiliar situations. The process begins with the initial confrontation of the problem and continues until an answer has been obtained and the learner has examined the solution. (Krulik, Rudnick & Milou, 2003:93).

Reys, Suydam & Lindquist (1984) give an analogy of problem-solving as “...involving a situation in which a person wants something but does not know immediately what to do to get it.” (p. 27) hence if it is immediately clear what to do to obtain what one wants, then there is no problem.

In essence problem-solving is not a simple matter of routine problems using readily available algorithms. One needs to think a while about what strategies to use to solve the problem. Hence cognition and higher order thinking are imperatives in the solution of problems. Higher order or critical thinking is an important element in enabling one to find a way of analysing the problem to find a solution path (Brumbagh & Rock, 2006).

Our stance on problem-solving is to recognise that there is a multiplicity of conceptions on it and various approaches in concurrence with Schoenfeld’s (1992) view that there exist contending and contradictory views of problem-solving. We accept his view that problem-solving should “train students to ‘think’ creatively and/or develop their problem-solving ability (with a focus on heuristic strategies)” (Schoenfeld, 1992, 10)

Polya (1945), in his seminal work, *How to solve it*, proposes four steps which one has to go through in the solution of a problem.

These steps are:

1. *Understanding the Problem:*

Understanding the problem ensures that the learner must first understand the verbal articulation of the problem and what it entails. This means that the student should be able to identify what is given and what is required.

2. *Devising a plan:*

This means that the learner must use what is given and think of possible strategies to implement towards the goal of solving the problem. The strategy may not be easy. Polya (1945) states that this “may be long and tortuous.” (p.8)

3. *Implementing the plan:*

This is self-explanatory. It means that the student having devised a plausible plan may then execute the solution plan.

4. *Looking back (Reflection):*

Looking back means that having solved the problem, the student has to look back to see if the solution is reasonable and correct because errors might have crept into the solution process.



## RESEARCH DESIGN

A qualitative research approach embedded in a constructivist paradigm was used in this study.

The data for this paper was obtained from a problem-solving exercise by 30 respondents comprising mathematics teachers and mathematics curriculum advisors held at our LEDIMTALI workshops in 2013. The participants were divided into groups and given approximately an hour to solve some trigonometric and financial mathematics problems and then present the group solution to all present. Subsequently, group discussed solutions were presented and motivated by representatives of each group. Each of these solutions was reflected upon and debated in terms of its mathematical correctness by all participants. The whole process from engaging with problems to exploring solutions up to their presentations and debates were video recorded.

This paper focuses on the following trigonometric problem because it involves proof which is an area of interest to the authors. The problem selected from Mathematical Digest.

Prove that in  $\triangle ABC$ ,  $\frac{\tan B}{\tan C} = \frac{a^2 + b^2 - c^2}{a^2 + c^2 - b^2}$  (Webb, Hardie, Allison, Brummer, Gilmour & Conradie, 1987)

It should be noted that in trigonometry, identities are proved in various ways. One method is to take the left hand side (or the right hand side) of the identity and simplify until it is identical to the right hand side (or the left hand side). Another method is to reduce both sides to a common expression or identity. It is important to note that during the proving process relevant trigonometric identities may be used to convert or simplify some of the identities to what is required to be proved. Sometimes reducing all or some of the identities or trigonometric ratios to sines and cosines would be an alternative way of proving.

In this paper we discuss three different ways of proving the identity presented by teachers in the workshop.

CASE 1

NO. 2.

$$\frac{\tan B}{\tan C} = \frac{a^2 + b^2 - c^2}{a^2 + c^2 - b^2}$$

$$\frac{\sin B}{\cos B} \div \frac{\sin C}{\cos C} = \frac{2ab \cos C}{2ac \cos B} \quad \begin{matrix} (B) \\ \text{Cosine} \\ \text{Rule} \end{matrix}$$

$$\frac{\sin B}{\cos B} \times \frac{\cos C}{\sin C} = \frac{b \cos C}{c \cos B}$$

Then By sin Rule:  $\frac{b}{\sin B} = \frac{c}{\sin C}$

$$\therefore \frac{b}{c} = \frac{\sin B}{\sin C}$$

$$\frac{\sin B}{\sin C} \times \frac{\cos C}{\cos B} = \frac{b \cos C}{c \cos B}$$

Looking at this solution one can observe that the teachers attempted to use the method of reducing the right hand side (RHS) and the left hand side simultaneously to the same expression, in this case,  $\frac{b \cos C}{c \cos B}$ . What we find in this solution is that the teachers invoked the sine rule, the cosine rule and the quotient identities for the tangent ratios. Although the structure of the proof is not in the conventional form, the group has brought to the proof salient ideas. For example it seems that the teachers have realised that they can transform the numerator and denominator of the expression on the RHS to incorporate a trigonometric ratio by applying the cosine rule to  $\triangle ABC$ .

In particular they have used the cosine rule to express the numerator  $a^2 + b^2 - c^2$  as  $2ab \cos C$  and the denominator  $a^2 + c^2 - b^2$  as  $2ac \cos B$  as shown in line 2 of the proof. Thereafter through algebraic simplification (i.e. cancellation of like factors), they reduced the quotient  $\frac{2ab \cos C}{2ac \cos B}$  to  $\frac{b \cos C}{c \cos B}$ .



It seems that group manipulated left hand side in the following sequence until they arrived at  $\frac{b\cos C}{c\cos B}$ :

- They use the tan quotient identity to express  $\tan B$  as  $\frac{\sin B}{\cos B}$  and  $\tan C$  as  $\frac{\sin C}{\cos C}$
- In step 3, it seems that the group realized that they could express  $\frac{\sin B}{\cos B} \times \frac{\cos C}{\sin C}$  as  $\frac{\sin B}{\sin C} \times \frac{\cos C}{\cos B}$
- Thereafter tried to show by using the sine rule that  $\frac{\sin B}{\sin C} = \frac{b}{c}$ .
- The teachers then substituted  $\frac{b}{c}$  for  $\frac{\sin B}{\sin C}$  with respect to the expression on

LHS of the identity in step3 to arrive at  $\frac{b\cos C}{c\cos B}$ , implying that the LHS= RHS, and thus completing the proof.

This solution was debated and considered correct except the expression of the solution with the equality sign from the very first statement indicating an assumption of the veracity of the identity at the commencement of the proof. A proper and acceptable proof in this instance would have been to take the right and left hand sides separately until each side is simplified to reach the result of  $\frac{b\cos C}{c\cos B}$ . However, we find that the group did not state that none of angles B and C can be  $90^\circ$  for a given  $\triangle ABC$ .

**CASE 2**

The second group presented the following proof:

$$\begin{aligned} \tan B &= \frac{\sin B}{\cos B} = \frac{b \cdot \sin C}{c} \times \frac{2ac}{a^2 + c^2 - b^2} \\ &= \frac{2ab \sin C}{a^2 + c^2 - b^2} \\ \tan C &= \frac{\sin C}{\cos C} = \frac{c \cdot \sin B}{b} \times \frac{2ab}{a^2 + b^2 - c^2} \\ &= \frac{2ac \sin B}{a^2 + b^2 - c^2} \\ \frac{\tan B}{\tan C} &= \frac{2ab \sin C}{a^2 + c^2 - b^2} \times \frac{a^2 + b^2 - c^2}{2ac \sin B} \\ &= \frac{b \cdot \sin C (a^2 + b^2 - c^2)}{c \cdot \sin B (a^2 + c^2 - b^2)} \\ \frac{\sin A}{a} &= \frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{or} \quad \sin B = \frac{b \cdot \sin C}{c} \\ \sin A &= \frac{a \sin B}{b} \quad \text{and} \quad \sin A = \frac{a \sin C}{c} \\ &= \frac{a^2 + b^2 - c^2}{a^2 + c^2 - b^2} \end{aligned}$$

The strategy adopted by this group was to work with the LHS of the identity and express  $\tan B$  and  $\tan C$  in terms of the sines and cosines of  $\hat{B}$  and  $\hat{C}$ . Now

$\tan B = \frac{\sin B}{\cos B}$  by using quotient identities for the tangent. Similarly  $\tan C$  is expressed in terms of the sine and cosine as  $\tan C = \frac{\sin C}{\cos C}$ .



The group now proceeds to use a combination of the sine and cosine formula for the expression  $\frac{\sin B}{\cos B}$ . Using the sine formula for the angles  $B$  and  $C$ , we have that

$\sin B = \frac{b \sin C}{c}$  and then using the cosine formula with angle  $B$  as a reference angle

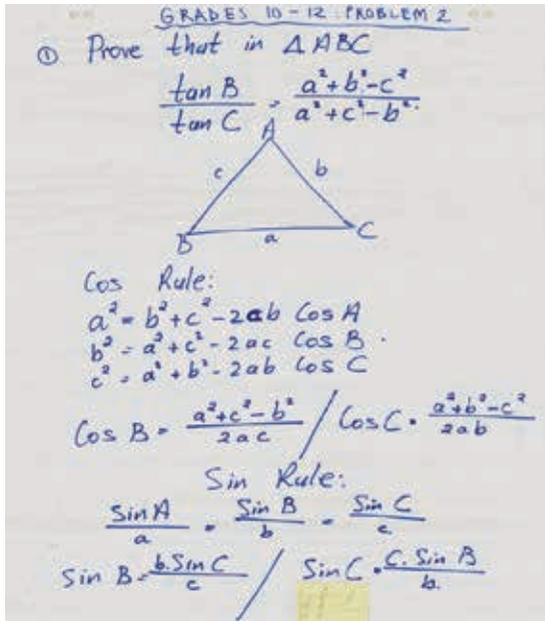
we have that  $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$ .

Through using algebraic manipulation, expressed  $\frac{\sin B}{\cos B}$  was written in the form  $\frac{b \sin C}{c} \times \frac{2ac}{a^2 + c^2 - b^2} = \frac{2ab \sin C}{a^2 + c^2 - b^2}$ . Similarly  $\tan C$  was expressed as  $\frac{2ab \sin B}{a^2 + b^2 - c^2}$ .

Thereafter, the group expressed  $\frac{\tan B}{\tan C} = \frac{2ab \sin C}{a^2 + c^2 - b^2} \times \frac{a^2 + b^2 - c^2}{2ac \sin B}$

Through cancellation of common factor 'a' in both the numerator and denominator, and having insight that  $b \sin C = C \sin B$  through using the sine rule, the group proceeded to write the express  $\frac{\tan B}{\tan C}$  as  $\frac{b \sin C (a^2 + b^2 - c^2)}{c \sin B (a^2 + c^2 - b^2)}$ . On the grounds that  $b \sin C = C \sin B$ , the group cancelled  $b \sin C$  in the numerator with  $C \sin B$  in the denominator. Hence, the group has shown that  $\frac{\tan B}{\tan C} = \frac{a^2 + b^2 - c^2}{a^2 + c^2 - b^2}$ . However, we find that the group did not state that none of angles  $B$  and  $C$  can be  $90^\circ$  for a given  $\triangle ABC$ .

**Case 3**



The last group presented the above incomplete proof. As can be observed above, it is clear that the group was able to identify the RHS as a quotient of expressions of the cosine formula. They began to express all sides of the triangle in terms of the cosine formula. They then made the cosines of angles B and C which are angles of the tangent quotient in the RHS of the identity to be proved. It seems that they were able to see the link between the tangents and the quotient ratios because they began to write down the sine formula. One hopes that if they had more time, they would be able to link the sine and sine formula like the first group and follow up to the proof.

## DISCUSSION

The proofs offered by both groups, demonstrate that they were able to read and understand the problem, and apply their minds to distil a strategy to solve the problem. Hence in this sense, it seems as if they followed Polya's problem-solving steps of initially reading to understand the problem, planning a strategy to tackle the problem and then solving the problem with a review at the end to check for correctness. Group 1 used the strategy of expressing the LHS and RHS to a common trig expression,  $\frac{b\cos C}{c\cos B}$ , through using quotient identities and the cosine formula coupled with algebraic manipulations. Group 2, started with less complex side (i.e. the LHS in this instance) and by using quotient identities, sine and cosine formula showed that the LHS = RHS. Unfortunately the third group did not finish the solution. One wonders why this group ran out of time compared to the other two groups. It might be a case of less experience teachers who did not receive assistance from their more experienced colleagues.

However, none of the groups took the more complex side (which is the RHS in this instance)  $\frac{a^2+b^2-c^2}{a^2+c^2-b^2}$ , and simplified to the LHS.

Referring to the "bottom lines" of success as articulated by Schoenfeld (2012), we can state that the teachers who are the subject of this paper were successful in as far as they were able to complete and present their solution, except of course the group who did not complete. The strategies employed by the teachers were clear and unambiguous. Unfortunately the teachers were not given time for metacognition or what is referred to by Schoenfeld (2012) as monitoring and self-regulation nor were they given time to discuss their experiences. Maybe it is the fault of the workshop facilitators.

It would also be interesting to see if one could construct a perpendicular from one vertex of the triangle to see if the usual right triangle definition of a tangent for the two angles B and C would yield the required identity. This exercise will also be followed up by interviews with some of the teachers to get a sense of their feeling about the process and their experiences.



## REFERENCES

- Brumbagh, D. & Rock, D. (2006). *Teaching Secondary Mathematics: Third Edition*. Mahwah, NJ.: Lawrence Erlbaum Associates, Publishers.
- Department of Basic Education. (2011). *Curriculum Assessment and Policy Statement: Mathematics (FET)*. Pretoria: Department of basic Education
- Horne, N. (1987). Readers' Problems. In J. H. Webb, K.A., Hardie, D. Allison, G.C.L
- Brummer, C.R.A., Gilmour & J.J. Conradie. (1987). *Mathematical Digest*. Cape Town: University of Cape Town. P.27.
- Jadama, L.M. (2014). Impact of Subject Matter Knowledge of a Teacher in Teaching and Learning Process. *Middle Eastern & African Journal of Research*. Issue 7. 20 – 29.
- Julie, C. (2011). *LEDIMTALI Brochure*. Bellville: University of the Western Cape.
- Krulick, S., Rudnick, J. & Milou, E. (2003). *Teaching Mathematics in Middle School: A Practical Guide*. Boston: Pearson Education. Inc.
- Lucus, C.A. (2006). Is subject matter knowledge affected by experience?: The case of composition of functions. In J. Novotna, M. Kratka. & N. Stehlikova. (Eds.). *Proceedings of the International Group for the Psychology of mathematics Education*. Vol. 4. Pp. 97 – 104. Prague: PME.
- Ma, L. (2010). *Knowing and Teaching Elementary Mathematics: Teachers' Understanding of Fundamental mathematics in China and the United States: Anniversary Edition*. New York: Francis & Taylor.
- National Council of Teachers of Mathematics. (2000). *Curriculum and Evaluation Standards for School Mathematics*. Reston, VA: Author.
- Norman, A. (1992). Teachers' mathematical knowledge of the concept of function. In G.Harel and E. Dubinsky. (Eds.). *The Concept of Function, Aspects of Epistemology and Pedagogy*. Washington DC: Mathematical Association of America. pp. 215 – 232.
- Polya, G. (1945). *How to Solve it: A new Aspect of Mathematical Method: Second Edition*. Princeton, NJ.: Princeton University Press.
- Reys, R.E., Suydam, M.N. & Lindquist, M. M. (1989). *Helping Children Learn mathematics: Second Edition*. Englewood Cliffs, New Jersey: Prentice Hall
- Schoenfeld, A.H. (1992). Learning to think mathematically: Problem-solving, metacognition, and sense-making in mathematics. In D. Gouws (Ed.), *Handbook for Research on Mathematics Teaching and Learning*. New York: MacMillan. 334 – 370.
- Schoenfeld, A.H. (2012). How we think: A theory of human decision-making, with a focus on teaching. 12<sup>th</sup> International Congress on mathematics Education. Seoul, Korea.
- Shulman, L. (1987). Knowledge and Teaching: Foundations of the New Reform. *Harvard Educational Review*. Vol. 57 No. 1. pp. 1 – 21
- Stein, M.K., Baxter, J. A. & Leinhardt, G. (1990). Subject matter knowledge and elementary instruction: A case from function and graphing. *American Educational Research Journal* 27, 639 – 663.

# GRADE 10 STUDENTS' FACILITY WITH RATIONAL ALGEBRAIC FRACTIONS IN HIGH STAKES EXAMINATION: OBSERVATIONS AND INTERPRETATIONS

<sup>1</sup>Duncan Mhakure, <sup>2</sup>Mark Jacobs and <sup>3</sup>Cyril Julie

<sup>1</sup>University of Cape Town, <sup>2</sup>Cape Peninsula University of Technology and

<sup>3</sup>University of Western Cape

*In this study students' facility with rational algebraic fractions is explored in a high stakes examination context. The presences of sub-constructs within fractions function as an additional complexity of rational fractions. We confirmed that the categories of student error that we found, mirrored a landmark previous study done by Figueras et al. (2008), but only in part; we found in addition that a number of students committed an "algebraic equation" error in that they converted rational fractions to algebraic equations and then tried to solve them. This study shows that the presence of visual cues in problems involving algebraic fractions act as distractive stimuli and we direct our analysis to these sub-constructs to deconstruct student difficulties with algebraic fractions. Our study makes useful recommendations for teachers and teaching.*

## INTRODUCTION

The teaching and learning of fractions is one of the most problematic areas in the Senior Phase (SP) and Further Education and Training Phase (FETP) school grades. In South Africa the SP refers to Grades 7, 8, and 9, whereas the FETP refers Grades 10, 11, and 12 (Department of Basic Education, 2011). Educational research regards fractions as a challenging concept in the curriculum, and alleges that "... problems in understanding fractions persist into adulthood, with moderate to severe consequences for everyday and occupational decision-making" (Ross & Bruce, 2009, p.713). During the past three decades, research in mathematics education has identified that the multifaceted nature of fractions is a major contributing factor to the core difficulties experienced by teachers and students during the teaching and learning of fractions. This multifaceted construct is made up of five interrelated sub-constructs as follows: *part-whole* (which gives rise to the notion of partitioning an object or set into smaller equal sections), *ratio* (gives the natural way of showing the procedures associated with finding the equivalent fractions), *operator* (this is key to developing an understanding of the multiplication operations of fractions), *measure* (refers to the idea that fractions are identified by their size, viz., the distance from a point of reference – this gives rise to the proficiency of ordering fractions on a number line), and *quotient* (underscores the notions that any fraction can be represented as a division operation, it is about fair-sharing, and the fact that the denominator is bigger than the numerator or vice versa does not matter). Learning fractions is difficult because it requires a deep understanding of all the latter sub-constructs. (Charalambous & Pitta-Pantazi, 2007; Behr et al., 1993; Kieren, 1993; Lamon, 1993, 1999, 2001; Marshall, 1993; Ross & Bruce, 2009).



Failure of students to conceptually understand these sub-constructs in the SP school grades leaves students with reduced chances for learning these skills in FETP school grades due to the congested nature of the curriculum. One key factor which contributes to the students' inability to learn fractions is the claim that teachers are not well equipped, through teacher education training programmes, to effectively teach the concept of fractions. As a result, rote learning appears to be the norm during teaching and learning of fractions (Gowan, et al., 1990). From a historical perspective, research has also shown that students in the SP and FETP school grades harbour a dislike of fractions primarily because they find "that fractions were irrelevant to the solution of mathematics problems in anyone's daily life" (Groff, 1996, p.177). In support for Groff (1996), Van Hiele (1986, p. 211) states "to be honest, we should admit that being able to calculate with fractions has no practical utility". These arguments put forward by Groff (1996) and Van Hiele (1986) support historical justifications for students to not engage with the learning of fractions.

Perceptions may, however, have changed during the past three decades (Usiskin, 2007). Usiskin (2007, p.370) underscores the importance and justification for inclusion of fractions in the formative years of mathematics education:

The realization that fractions represent division and constitute the most common way in which division is represented in algebra has caused a demand for increasing competence in fractions by all those for whom algebra skills are important.

Whilst there is a common agreement that fractions are difficult for students to conceptualise, there is generally strong support for the critical importance of teaching fractions to SP students. The justifications for doing so are many. Firstly, formal and prolonged exposure to the teaching and learning of fractions equip students with the pre-requisite necessary skills for "partitioning" that will enable them to understand and describe real world phenomena (Groff, 2006). Secondly, fractions are the gate for higher mathematics – this means that if students have a weak background in fractions they are likely to experience challenges in understanding concepts in algebra, higher-order mathematics such as number theory, and calculus (Aliberti, 1981; Karim et al., 2010). Brown & Quinn (2007b, p.9) claim that "to solve rational equations and simplify rational fractions it is necessary to apply generalised common fractions concepts". The point here is that students whose understanding of fractional constructs are weak are likely to find rational algebraic fractions concepts difficult (Brown & Quinn, 2006, 2007a, 2007b; Usiskin, 2007; Groff, 2006). In contrast, not all mathematicians and educators are convinced about the intrinsic value of fractions when solving real life problems or as the pre-requisite for higher order mathematics. The quotient sub-construct, argue Enright (1998) and Van Hiele (1986), is difficult to implement because dividing objects in perfectly equal parts is practically impossible.

Hence, teachers will find it difficult “to make up genuine problems that can be solved through the manipulation of fractions” (Groff, 2006, p.552). Figueras et al. (2008) suggest that rational fractions pose another particular problem: it is difficult to give concrete examples to go with the rational algebraic fractions introduced, consequently students may also struggle to generate examples themselves.

Whilst we acknowledge the concerns raised in the rebuttals that proficiency in fractions does not translate to successful study of algebra, in this article we argue that students with insufficient knowledge of fractional sub-constructs from the SP school grades are more likely to experience learning difficulties when simplifying rational algebraic fractions they encounter in the FETP curriculum.

Rational algebraic fractions contain all the difficulties of ordinary fractions, but in addition it includes those difficulties usually associated with algebra generally. These include addition and subtraction involving like and unlike terms, multiplication and division. These operations take place *inside* each algebraic fraction and thus mirror some of the sub-constructs of ordinary fractions. Further, other difficulties include the addition of algebraic fractions, the use of a lowest common denominator and the operations involved in adding these terms, in much the same way that ordinary fractions are added.

### CLASSIFICATION OF ERRORS ON SIMPLIFYING RATIONAL EXPRESSIONS

In addition, and acting in a complementary fashion to the idea of sub-constructs, Figueras et al. (2008), found seven categories of student error when testing for students’ understanding of simplifying rational expressions. They administered the tests after the students had specifically been taught rational expressions and arrived at the following categories: (a) *Cancellation* – using a constant term, variable, coefficient that was present in both numerator and denominator and was cancelled. For example, if we consider the category “cancellation” we may observe that students usually do not understand that in any fraction if the denominator and numerator are at the same time multiplied or divided by the same number, the original fraction does not change.

For example:  $\frac{2(5)}{3(5)}$ ,  $\frac{8 \div 4}{12 \div 4}$ , and  $\frac{2(3x+4)}{2(x+y)}$  will be the same as  $\frac{2}{3}$ ,  $\frac{8}{12}$  and  $\frac{3x+4}{x+y}$

respectively. Secondly, however, if the same number is added or subtracted to both the numerator and denominator of any fraction, the resulting fraction will be different for the original one. For example:  $\frac{5+3}{7+3}$ ,  $\frac{5-2}{8-2}$ , and  $\frac{4a+3}{4b+1}$  will not be the same as  $\frac{5}{7}$ ,

$\frac{5}{8}$  and  $\frac{a}{b}$  respectively.



(b) *Partial division*: division taking place but only between some of the terms. (c) *Like term error 1* - students committing an error other than division (usually subtraction) between like terms in the numerator and denominator. (d) *Like term error 2* - performing a mistaken operation in the numerator or denominator (for example adding wrong). *Linearization* - breaking up a rational expression with a compound into two separate rational expressions. (e) *Defractionalisation* - transformation of fraction with unity numerator to a non-fraction, for instance,  $\frac{1}{3}$  becomes 3. (f) *Equationisation* - transforming a rational fraction into a rational equation.

The categories “like term error (1) and (2)” above can clearly be further delineated into still more categories if one chooses. For example, one can distinguish between the “mistaken operations” which students make in the numerator and denominator, or for that matter in the *kinds* of addition which students were making which were wrong. There is also a conflation of addition and multiplication when working with rational fractions, much like the conflation which happens when working with exponents. These categories sound more like the sub-construct of fractions discussed above.

Thus underlying the concept of sub-constructs and also the prevalence of particular forms of errors are operations and understanding of basic algebra, involving addition and subtraction of like and unlike terms, multiplication and division of algebraic terms and so on. Therefore, as other researchers concur, the area of simplifying rational expressions in algebra research is a resource with a rich supply of data as we also experienced. It is claimed by some that these kinds of errors are not focussed on by teachers because they do not realise that students struggle to cope with the fundamental concept of cancellation, for example, among other difficulties. In fact it is claimed that student apply rote learning when simplifying fractions and rational algebraic fractions as a matter of course (Grossman, 1924).

### PURPOSE OF THE STUDY

One of the key objectives of learning algebra at the key FETP in Grade 10 is to manipulate algebraic expressions by: “Simplifying, adding and subtracting algebraic fractions with denominators of cubes (limited to sum and difference of cubes)” (Department of Basic Education, 2011, p.13). Drawing on the theoretical model of the five sub-constructs on fractions and rational numbers presented by Charalambous and Pitta-Pantazi (2007), and the categories of errors as outlined by Figuera, et al. (2008), this study investigated the proficiencies of grade 10 mathematics learners in simplifying rational algebraic fractions in a high-stakes end of year examination. In particular, the study sought answers to the following research question: How proficient are grade 10 mathematics learners in the simplification of rational algebraic expressions? The findings of this study will serve as a platform for further research that will provide insights into high school mathematics students’ understanding of simplifying rational algebraic fractions.

In addition, the study can inform classroom practice of mathematics teachers, teacher educators of mathematics, and teachers of post-secondary school mathematics on the teaching and learning of rational algebraic fractions. It is critical for students to have an understanding of rational expressions, especially students who intend to successfully study majors in science, technology, engineering, and mathematics (STEM).

## METHODS

The data collected consists of end of year test scripts of the grade ten mathematics classes in schools involved in the Local Evidence Driven Improvement Mathematics Teaching and Learning Initiative (LEDIMTALI) Project. This project, a First Rand Mathematics Chair project based at University of Western Cape, involves: inservicing mathematics teachers from several schools in a teacher development programme, improving understanding and teaching of mathematics in schools. In particular, student responses to question 2.3 of the examination paper is analysed in this study.

The problem (question 2.3) given to the students contained three rational fractions, namely,

$$(a) \frac{x}{x+y}, (b) \frac{x^2+y^2}{y^2-x^2}, (c) \frac{2x+y}{x+y}.$$

The question was to prove that:  $\frac{x}{x+y} - \frac{x^2+y^2}{y^2-x^2} = \frac{2x+y}{x+y}$ . There was a typographical

error in fraction (b) as it was supposed to read:  $\frac{x^2-y^2}{y^2-x^2}$ , however, this typographical

error does not impact on the overall results of this study whose main focus was not to assess whether students can prove the equivalence of the rational algebraic fractions, but to show students' proficiencies in simplifying rational algebraic fractions.

In all three algebraic fractions, the *visual cues* provided tempting examples of like terms in the denominator and numerator which could be cancelled. In fraction (a), the denominator contains a sum of two unlike terms, and the visual cue of an  $x$  in the numerator and denominator. A student could thus decide to add the terms in the denominator, or cancel the  $x$ 's. In fraction (b) there are two sub-constructs, an addition of two squares in the numerator and the difference between two squares in the denominator. The visual cues of the  $x$ 's and  $y$ 's may cause the student to use the cancellation error twice. Or the student may recognize the case of the two squares and factorise (but factorising the sum of squares would be seen as an error of factorization). In the third case (c), there are again two sub constructs, both sums of unlike terms. Again the visual cues of the presence of  $x$ 's and  $y$ 's may lead to the cancellation error.



The overall picture (did the student see the wood as well as the trees?) is the requirement to prove that (a) minus (b) equals (c). The student must perform a subtraction (a minus b), *simplify* the quotient and *show* that this is equal to the third algebraic fraction, (c).

Thus the problem contains a myriad of operations, any number of which can go wrong. With this kind of procedural problem is that it is very difficult to determine why things went wrong if they do go wrong. In such cases we must make speculative judgement calls. Diagnoses require detailed exploration of all possible options.

### DATA ANALYSIS

From an initial sample of 39 students (that is, one batch containing students from one school) we created the following categories: 14 simply wrote down the question, and there was no attempt to solve it, 15 left out the question (did not write it down). We were left to analyse 10 scripts. The sample of ten scripts was further broken down as follows: four were analysed to assess the extent to which lack of facility with sub-constructs led to errors in solving the problem posed. All ten were analysed in terms of the categories of errors. As it turned out, the use of the cancellation error was the most frequent in the four sample scripts analysed for sub-construct errors.

#### Sub-constructs

2.3  $\frac{x^2 + y^2}{xy - y^2} = \frac{2x + y}{xy}$  → Line 1

$\frac{x^2 - y^2}{x^2 + y^2} = \frac{2x + y}{xy} \therefore y = 2$  → Line 2

Figure 1: Sub-construct and cancellation error – example 1

In figure 1, the student applied the cancellation error thrice and was left with: a y in the denominator, a 2 in the numerator in the third algebraic fraction and numerous negative signs (notice there was no quotient “1” after “like” terms have been “cancelled”). The student then ends up with the solution:  $y=2$ . The *visual cues* appeared to have dominated this student’s approach to solving the problem. There was no attempt to deal with the sub-constructs.

2.3.  $\frac{2x}{x+y} = \frac{2x^2+y^2}{y^2-x^2} = \frac{2x+y}{x+y} \rightarrow$  Line 1  
 $\frac{2x}{x+y} = \frac{1}{1} = \frac{2x+y}{x+y} \rightarrow$  Line 2

Figure 2: Sub-construct and cancellation error – example 2

In figure 2, the student applied the cancellation error to the second algebraic fraction only, was left with an answer for that operation, namely 1, and left the problem in that shape. For reasons that can only be brought out during an interview, the student did not cancel any of the other terms. Again no attempt was made to engage the sub-constructs.

In figure 3, the student correctly joins the two algebraic fractions (thereby demonstrating understanding for addition of algebraic fractions), and, at the same time ends up with a new algebraic fraction on the right hand side, notably with a denominator equal to that on the left hand side. An alternative route for the student at that point was to equate the two numerators, especially after having made the denominators equal, presumably for that purpose.

2.3.  $\frac{x}{x+y} - \frac{x^2+y^2}{y^2-x^2} = \frac{2x+y}{x+y} \rightarrow$  Line 1  
 $LCD = y^2 - x^2$   
 $\frac{y-x(x) - x^2+y^2}{y^2-x^2} = \frac{2x^2+y^2}{y^2-x^2} \rightarrow$  Line 2  
 $\frac{-x^2+xy - x^2+y^2}{y^2-x^2} = \frac{2y^2+y^2}{y^2-x^2} \rightarrow$  Line 3  
 $\frac{-2x^2+xy+y^2}{y^2-x^2} = \frac{2+1}{1-1} \rightarrow$  Line 4  
 $\frac{-2+xy}{1-1} = \frac{3}{c} \rightarrow$  Line 5  
 $\frac{2+xy}{c} = \frac{3}{c} \rightarrow$  Line 6

Figure 3: Sub-construct and cancellation error – example 3

Again further probing of the student may provide more information. Extensive use is made of the sub constructs to factorise and add algebraic fractions using a lowest common multiple approach.



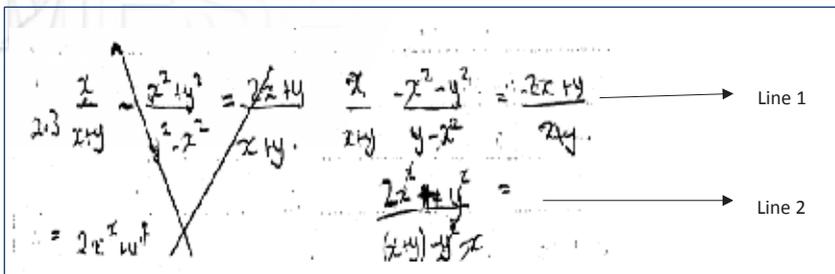


Figure 4: Sub-construct and cancellation error – example 4

In Figure 4, there is an example of the type of error which has been called type 2 (T2) in Figueras, et al. (2008). From the point of view of sub-constructs it is not clear from the script what methods the student employed to bring the two algebraic fractions together. This is then followed by cancellation.

**CATEGORIES OF TYPES OF ERRORS**

In this section we discuss four categories of the errors found in the ten scripts that we analysed, and how they manifested in the responses. These are: “cancellation” errors, “performing a mistaken operation” error, “grouping like terms” error, and “factorisation” errors. The “performing a mistaken operation” (T2) error fits better in the “errors committed in the sub-constructs” as detailed above, but are repeated here, as part of the original classification of Figueras, et al. (2008).

From the ten students whose scripts were analysed we observed that: 4 committed the T2 error Figueras et al. (2008) performing a mistaken operation in the numerator and/or denominator, 5 had cancellation errors (this included other errors not classified by the Figueras et al. (2008), 2 had a *factorisation* error (they responded the same way) but cancelled properly afterwards, and 4 committed what we termed an algebraic equation error – they converted the rational expressions into an algebraic equation (varied forms).

**Cancellation errors**

Typically, the cancellation error involved students cancelling like terms in the numerator and denominator, without consideration for the rational fraction as a whole, much as was documented by Figueras et al. (2008). This was covered in the sub-constructs section.

**Performing a mistaken operation error**

The “like term error 2” which involved mistakes in which students simply joined variables as in  $2x + y = 2xy$  treating the expression  $2x + y$  within the wider problem as if it was a multiplication problem, occurred a number of times. For examples of this type of error see figure 5, lines 2.

1.3  $\frac{x}{x+y} - \frac{x^2+y^2}{x^2-x^2} = \frac{2x+y}{x+y}$  → Line 1

$\frac{x}{x+y} + \frac{y^2}{x^2} = \frac{2x+y}{x+y}$  → Line 2

$\frac{x(x^2) + (y^2)(x+y)}{(x+y)(x^2)}$  → Line 3

$= \frac{x^3 + x^2y + xy^2 + y^3}{x^2(x+y)}$  → Line 4

$= \frac{x^2 + xy^2}{x^2y}$  → Line 5

$= \frac{x^2+y^2}{xy}$  → Line 6

Figure 5: Mistaken operation error

The sub-text of student responses here appears to be that the student knows to bring all the rational fractions to a state where they could be joined under one denominator and then calculated – the students certainly aimed for that. Further errors occur once there is the comfort of joining the fractions under one denominator – for example calculation errors classified in type 1. In the main, once the denominator is created (and is false) the students’ work is so wrong that it cannot be saved by another contrived step.

### Grouping like terms error

Of particular interest, and not mentioned in the Figueras et al. (2008) list is the students’ use of an algebraic equation in place of the rational fraction, which is then solved.

2.3.  $\frac{x}{x+y} - \frac{x^2+y^2}{y^2-x^2} = \frac{2x+y}{x+y}$  → Line 1

$\frac{x}{x+y} + \frac{y^2}{y^2-x^2} = \frac{2x+y}{x+y}$  → Line 2

$= x+x^2=2x$  and  $y^2=y \Rightarrow \frac{2x+y}{x+y}$  → Line 3

then  $= y^2-x^2=0$  and  $x+y \Rightarrow x+y$

Figure 6: Grouping like terms error

There is either logic, which on the face of it has no sense (but is logical anyway) (see line 2, when each part of one numerator on the left hand side is “added” and then equated to the numerator in the other side), while some mixed logic is applied to the denominator (line 3:  $y^2 - x^2$  is equated to  $x$  plus  $y$ , both groups in denominators of their respective fractions).



### Factorisation errors

The existence of factorisation errors (figure 7) meant that at least some students thought that they had to factorise to solve the problem. The difficulty of the second rational fraction is that  $y^2 - x^2$  is an example of the difference between two squares and can be factorised using a formula,  $y^2 + x^2$  could not be factorised. However the *visual cue* in this case may have been that the two expressions were placed in a rational fraction – thus *the temptation to cancel* was presented, but only once the students had factorised. *Is this a case where knowledge of the outcome leads to certain forms of behaviour?* In both cases the students factorised that rational fraction in order to cancel like terms. The cancellation was proper but the factorisation was not. This indicated that these two students knew what kind of cancellation was not allowed and what kind was *desirable*.

2.3  $\frac{x}{x+y} - \frac{x^2+y^2}{y^2-x^2} = \frac{2x+y}{x+y}$  Line 1

$\frac{x}{x+y} - \frac{(x-y)(x+y)}{(y-x)(y+x)} = \frac{2x+y}{x+y}$  Line 2

$\frac{x}{x+y} - \frac{x(y)}{(x/y)(x+y)} = \frac{2x+y}{x+y}$  Line 3

$\frac{x}{x+y} = \frac{2x+y}{x+y}$  Line 4

Figure 7: Factorisation error

### DISCUSSION OF THE FINDINGS

In reviewing the overall mathematical behaviour of the students who attempted the question under discussion, a number of issues lend themselves to speculation. For example, what is the role of the right hand side in questions in which students have to prove an equation or identity? In this case, the students had to prove that a left hand side, which was hugely complicated, equalled a right hand side, which looked more manageable. Some were clearly totally out of their depth and simply left the question out or wrote it down but did not attempt it. Of particular note is that one student scored in the top percentile for the test and yet left out question 2.3. There is no evidence that the student attempted the problem and discovered that it could not be solved.

Our assumption is that the student found the problem to be too difficult to attempt in the examination context. Others attempted to deal with the complications which the left hand side of 2.3 offered, using a number of documented error-strewn strategies, to little or no effect, but always striving to bring or force the left hand side to be the same as the right. In a few cases, the students attempted to “solve” the equation, providing a value as solution for  $x$  or  $y$ .

There is some research which shows that certain students are not ready to accept arithmetic expressions as concepts in their own right, for example  $2+3$ ; that they expect that such a concept (“sum”) is actually a process (addition) which has as its end goal a mathematical object, namely 5 (which is at the same time a mathematical *symbol!*). Gray and Tall (1994) introduced the notion of a *precept*, that is, a mathematical object that was at once a process and a concept. They postulated that how the students viewed the mathematical object would influence what they would try to do with it. Translated to algebraic expressions, the algebraic fraction  $\frac{x}{x+y}$  is at

once an algebraic fraction  $\frac{A}{B}$  as well as, in the denominator, an algebraic expression.

We thus have a precept within a precept: the algebraic fraction consisting of two separate expressions  $A$  and  $B$ , and, in the case of  $B$ , another precept (addition of  $x$  and  $y$  and the sum “ $x+y$ ”). The result is that some students may lean towards a procedural way of thinking about mathematics while others will develop a perceptual way of thinking about mathematical problems. Solving complex mathematical structures such as algebraic fractions can be a dividing line for these different levels of thinking about and doing mathematics.

## RECOMMENDATIONS

It is clear that algebraic fractions are multi-complex and contain a myriad of difficulties for students in the early stages of learning algebra. The existence of several sub constructs within any one construct of an algebraic fraction and the difficulties of seeing algebraic fractions as entities in their own right (precepts) and not only as procedures all impact on students’ abilities to navigate procedural mathematical problems.

For diagnostic purposes, it is perhaps better suited to use this kind of problem with a set (group) for whom basic operations in algebra has been achieved at the required level of competency. The concatenation of levels of algebraic development cannot be very useful in a developmental context.

We questioned whether the cases where knowledge of the end result lead to certain forms of behaviour, especially where students have to prove that one side of an equation equals another. This could be further explored.



## CONCLUSION

In conclusion, the findings of this study have implications for the teaching and learning of rational algebraic fractions in secondary schools. This study has demonstrated the existence and the extent of students' conceptions, and misconceptions related to rational algebraic fractions. The nature of the students' difficulties with respect to rational algebraic fractions is conceptual – students may be operating on rational algebraic fractions without necessarily understanding or justifying what they are doing. Given that the current study focussed on the analysis of the examination scripts, we posit here that additional research, that includes the qualitative analysis of students' written examination scripts and tasks-based focus groups interviews, is required to gain further insights into the students' deficiencies with rational algebraic fractions. Finally, we are of the opinion that if understanding rational algebraic fractions is fundamental to successful studying of advanced mathematical concepts required in STEM, then deficiencies in rational algebra fractions need to be identified and addressed early.

## REFERENCES

- Aliberti, A. J. (1981). Fractions: An endangered number form. *Curriculum Review*, 20, 374 – 381.
- Behr, M.J., Harel, G., Post, T., & Lesh, R. (1993). Rational numbers: Toward a semantic analysis-emphasis on the operator construct. In T. P. Carpenter, E. Fennema, & T. A. Romberg (eds.), *Rational numbers: An Integration of Research*, Lawrence Erlbaum Associates, NJ, pp. 13-47.
- Brown, G. & Quinn, R. J. (2006). Algebra students' difficulty with fractions: An error analysis. *The Australian Mathematics Teacher*, 62(4), 28-40.
- Brown, G. & Quinn, R. J. (2007a). Fraction proficiency and success in algebra: What does research say? *The Australian Mathematics Teacher*, 63(3), 23-30.
- Brown, G., & Quinn, R. J. (2007b). Investigating the relationship between fraction proficiency and success in algebra. *The Australian Mathematics Teacher*, 63 (4), 8-15.
- Charalambous, C. Y. & Pitta-Pantazi, D. (2007). Drawing on a theoretical model to study students' understanding of fractions. *Educational Studies in Mathematics*, 64: 293-316.
- Department of Basic Education. (2011). National Curriculum Statement – Curriculum and assessment Policy Statement (Mathematics). Further Education and Training Phase Grades 10, 11 and 12. *Government Printing Works*. Pretoria. South Africa.
- Enright, B. E. (1989). *Basic mathematics: Detecting and correcting special needs*. In J. S. Choate (Ed.). Boston: Allyn and Bacon, pp. 106.
- Figueras, H., Males, L., & Otten, S. (2008). *Algebra Students' simplification of rational expressions*. Retrieved 3/3/2014 from <http://www.msu.edu/ottensam/RationalExpressionSimplification.pdf>

- Groff, P. (1996). Is teaching fractions a waste of time? *The Clearing House*, 69(3), 177-179.
- Grossman, A. (1924). An analysis of the teaching of cancellation in algebraic fractions. *Mathematics Teacher*, 17 (2), pp.104-109.
- Karim, M., Leisher, D., & Liu, C. (2010). Pre-Calculus Algebra: Original Purpose—Its Relevancy to Students’ Needs—and Some Suggestions. *Atlas Journal of Science Education*, 1(2), 19-23.
- Kieren, T. E. (1993). Rational and fractional numbers: From quotient fields to recursive understanding. In T. P. Carpenter, E. Fennema, & T. A. Romberg (eds.), *Rational numbers: An Integration of Research*, Lawrence Erlbaum Associates, NJ, pp. 49-84.
- Lamon, S. J. (1993). Ratio and proportion: Children’s cognitive and metacognitive process. In T. P. Carpenter, E. Fennema, & T. A. Romberg, (eds.), *Rational Numbers: An Integration of Research*, Lawrence Erlbaum Associates, New Jersey, pp. 131-156.
- Lamon, S. J. (1999). *Teaching fractions and ratios for understanding*. Lawrence Erlbaum Associates, New Jersey.
- Lamon, S. J. (2001). Presenting and representing: From fractions to rational numbers. In A. Cuoco and F. Curcio (eds). *The roles of representations in school mathematics – 2001 Yearbook*, Reston: NCTM, pp. 146-165.
- Marshall, S. P. (1993). Assessment of rational number understanding: A schema-based approach. In T. P. Carpenter, E. Fennema, & T. A. Romberg (eds.), *Rational Numbers: An Integration of Research*, Lawrence Erlbaum Associates, New Jersey, pp. 261-288.
- Ross, J. A. & Bruce, C. D. (2009). Student achievement effects of technology-supported remediation of understanding of fractions. *International Journal of Mathematics Education in Science and Technology*, 40(6), 713-727.
- Usiskin, Z. P. (2007). Some thought about fractions. *Mathematics Teaching in the Middle school*, 12 (7), 370-373.
- Van Hiele, P. M. (1986). *Structure and thought: A theory of mathematics education*. Orlando, Fla. Academic.



## DIAGRAMS: ARE THEY USEFUL IN MATHEMATICS?

Vimolan Mudaly

University of KwaZulu-Natal

### *Extended Abstract*

This presentation describes the use of diagrams as self-explanatory tools. It considers the use of diagrams in general, and more specifically, it considers some research that is currently being undertaken in the broad field of visualisation. The research participants in this paper were Advanced Certificate of Education students and primary school learners. Further, the paper attempts to very briefly analyse their responses to some questions that they had to answer. The outcome of this research is pointing towards the strategic use of diagrams when dealing with problem solving. This is an ongoing research but the paper attempts to capture the current status of the research.

# MATHEMATICS TEXTBOOK ANALYSIS: A GUIDE TO CHOOSING THE APPROPRIATE MATHEMATICS TEXTBOOK

Benadette Aineamani & Seshni Naicker

Pearson Holdings South Africa

*There is a lot of research which has been carried out on the influence of mathematical textbooks on student teaching and learning of mathematics (Morgan, 1995; 2005; Dowling, 1996). Since the mathematics textbook has such a pivotal role within the classroom, the impact of the mathematics textbook on student learning is undeniable. This research, by means of mathematics textbook analysis, investigates the effectiveness of the Pearson mathematics textbooks for developing student comprehension and motivation. In order to analyse the textbooks, we have used two theoretical frameworks to analyse Pearson mathematics textbooks. The theoretical frameworks used are Kilpatrick et al's (2001) five strands of mathematics proficiency and Marton et al's (2004) Variation theory. From the two frameworks, we developed an analytical framework, which we used to analyse the textbooks. Conclusions from the textbook analysis are that the textbook used in the classroom is very important in determining how learners view mathematics, the textbook helps in enabling or restricting learners to communicate their mathematics ideas and also the textbook should probe learners for higher mathematical reasoning by asking questions that require them to give more explanation and good justifications for their responses in the mathematics classroom.*

## INTRODUCTION

Mathematics is one of humanity's great achievements (Mckenzie, 2001). In order for people to participate fully in society, they must know basic mathematics. Innumeracy deprives them not only of opportunity but also of competence in everyday tasks, for example banking and shopping. Citizens who cannot reason mathematically are cut off from whole realms of human endeavour (Kilpatrick et al., 2001). Mathematics is an intellectual achievement of great sophistication and beauty that uses the power of deductive reasoning (Muller & Maher, 2009). Deductive reasoning is a logical process in which a conclusion is based on the accordance of multiple premises that are generally assumed to be true.

The mathematics students need to learn today is not the same mathematics that their parents and grandparents needed to learn (Bauer, 2013). When today's students become adults, they will face new demands for mathematics proficiency that school mathematics needs to attempt to anticipate. The mathematics textbook which students use should look at all aspects in an attempt to anticipate the needs of the students such as introducing deductive reasoning explicitly so that learn how to use the reasoning in mathematics (Muller & Maher, 2009).

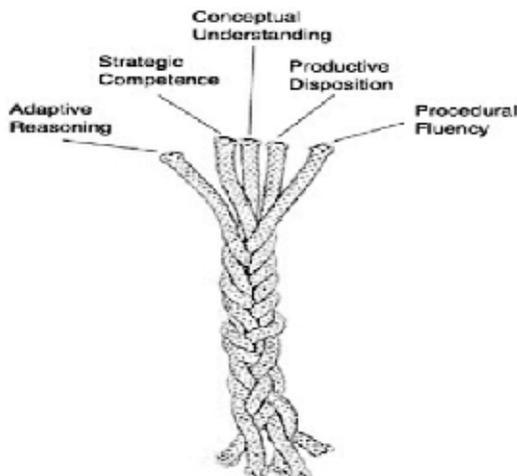


According to the Curriculum, there are four cognitive levels in mathematics, namely: Knowledge, Routine Procedures, Complex Procedures and Problem solving. In this study, we have used two theoretical frameworks to analyse Pearson mathematics textbooks. The theoretical frameworks used are Kilpatrick et al's (2001) five strands of mathematics proficiency and Marton et al's (2004) Variation theory.

### **KILPATRICK ET AL'S FIVE STRANDS**

Kilpatrick, Swafford and Findell (2001) conducted research in mathematics classrooms in the US, over a long period of time and they came up with a theory that in order for a learner to successfully learn mathematics, five strands of mathematics proficiency should be developed: Conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition. Procedural fluency refers to the skill in carrying out procedures flexibly, accurately and appropriately (Kilpatrick et al., 2001). Strategic competence refers to the ability to formulate, represent, and solve mathematical problems while Adaptive reasoning refers to the capacity for logical thought, reflection, explanation, and justification (Kilpatrick et al., 2001). Productive disposition refers to the ability to 'see' mathematics as sensible, useful, and worthwhile (Kilpatrick et al, 2001). Conceptual understanding enables learners to learn new ideas by connecting those ideas to what they already know. It supports retention and also prevents common errors (Kilpatrick et al., 2001). Knowledge learned with understanding provides a foundation for generating new knowledge and for solving unfamiliar problems (Bransford et al., 1999).

The five strands are interwoven as shown in the diagram below and hence development of one strand depends on development of the other strands (Kilpatrick et al., 2001: 116).



Kilpatrick et al's (2001) theory argues for learning with understanding because it is more powerful than simply memorizing and also organisation improves retention, promotes fluency and facilitates learning related material. Having a deep understanding requires that learners connect pieces of knowledge, and that connection in turn is a key factor in whether they can use what they know productively in solving problems. Kilpatrick et al (2001) argue that their theory can be adapted to any mathematics teaching and learning environment.

## VARIATION THEORY

Variation theory is a theory about learning (Runesson, 2005). Learning is defined as the "process of becoming capable of doing something as a result of having had certain experiences" (Marton, Runesson and Tsui, 2004: 5). "Learning always has an object" (Runesson, 2006: 406) because there is always something that has to be learnt, and once the critical features of what is being learnt are discerned, then learning has taken place (Runesson, 2006). The object is defined as "that which is the focus of attention" (Watson and Mason, 2006: 100). Marton et al. (2004) and Runesson (2005/ 2006) give a detailed discussion about variation theory. Variation theory focuses on the differences in the dimension or values of a feature. According to variation theory, it is not possible to discern a certain way of thinking about something without the contrast of other ways of thinking about the same thing (Marton et al. 2004). It is important to note that Marton et al. (2004) say that they are not advocating for an overall application of variation theory but they believe that variation theory can be used to enable learners to experience the features that are critical for particular learning and also to help learners in developing certain capabilities. Marton et al. (2004) define capabilities as the object of learning. The object of learning is defined by its critical features and the critical features are those features that should be discerned in order for the meaning that is aimed for to be understood. Mathematics as a subject has objects of learning, for example, algebra as a topic is referred to as an object of learning. The object of learning has the *general* and the *specific* component. The general component involves the nature of the capability, for example, the general component of algebra involves remembering, interpreting, discerning and grasping the various features within algebra. The specific component involves the actual 'thing' to which the acts of remembering, discerning and grasping are carried out (Marton et al, 2004). The specific object of learning is also referred to as the direct object of learning while the general object of learning is referred to as the indirect object of learning (Marton et al, 2004).

According to variation theory, careful attention needs to be paid in respect to what is varying and what is invariant in any learning situation in order to understand "what it is possible to learn in that situation and what not" (Marton et al, 2004: 16). There are different patterns of variation as discussed by Marton et al. (2004). These are contrast, generalisation, separation and fusion.



### a) Contrast

In order for the learners to discern critical features of a given object of learning, they have to experience something that is not a critical feature of that object of learning so that they compare the different features (Marton et al, 2004). Runesson (2005) argues that a learner is likely to experience something if it is contrasted with “what it is not” (Runesson, 2005: 84), for example, a learner is able to experience  $4x+3$  as an expression if he is given  $4x+3=8$  which is not an expression, but an equation. Simply pointing out the features of an algebraic expression is not enough, therefore, something that is not an algebraic expression must be shown to the learners so that they have something to contrast with.

### b) Generalisation

In order for learners to experience critical features of a given object of learning, various examples showing the critical features of the object of learning should be given to the learners and this brings out the dimension of the varying aspects of the features of the object of learning (Runesson, 2005), for example, different algebraic expressions such as  $x+y$ ,  $m+4x$ ,  $y+5x+9$ ,  $-3x+7g$ , and  $5x+4y-2x$  should be given to the learners so that they can see several examples of how algebraic expressions look like in order to discern critical aspects of algebraic expression.

### c) Separation

In order to experience a certain aspect of the object of learning, the aspect should be separated from other aspects by varying that particular aspect and keeping the other aspects invariant (Marton et al, 2004), and this is referred to as controlled variation (Watson and Mason, 2006). For example, in order for the aspect of being able to use various letters in an algebraic expression to be experienced, learners can an expression:  $x+y$ ,  $m+n$ ,  $z+u$ , whereby the structural meaning of the expression  $x+y$  is invariant but the letters are changing. By giving the learners such expressions, they are able to discern the feature that any letter can be used in an algebraic expression without actually changing the meaning of that expression. Learners discern differences between and within objects through attending to variation (Watson and Mason, 2006). Variation must be controlled because if everything is varying at the same time, nothing maybe discerned., for example, expressions such as  $x+y$ ,  $2x-m+u$ ,  $n+2m$  are given for learners to discern the critical feature of being able to use various letters in algebra, many things are changing in these expressions such as the coefficients, the structure and the number of variables in the expressions. Therefore, it may be difficult for the learners to discern the critical feature being put across. In other words, random variation does not help learners to focus their attention on the critical features of the object of learning (Watson and Mason, 2006).

### **d) Fusion**

Learners need to be able to discern different critical features of the object of learning at the same time (Runesson, 2005), and this is referred to as synchronic simultaneity (Marton et al, 2004). Discerning simultaneously is experienced against the background of previous experience (Marton et al, 2004), for example, within an expression such as  $4m+x-2z$ , the learner should be able to discern that there are three different variables, with different coefficient and one of the variables has a coefficient of one. If the learner is able to discern all the features at the same time, then this is known as synchronic simultaneity.

### **WHY TEXTBOOK ANALYSIS?**

Most learners solve mathematics problems mindlessly and they try to follow the textbook examples without understanding (Mckenzie, 2001). Some learners think that there are easy and hard ways of solving mathematics problems (absence of strategic competence) (Aineamani, 2010). They tend to ask their teachers for easier ways other than what is given in the textbook. There is a gap in some learners' reasoning and communicating mathematically when they move from written to spoken mathematics. Some learners' adaptive reasoning is not well developed because they cannot explain and justify what they have written down (Aineamani, 2010). Therefore, the textbook which the learners uses should have activities that help the learners develop the five strands of mathematics proficiency.

The textbook should be written in such a way that mathematics concepts are well explained to learners even in the absence of the teacher. Teachers should only help learners with mathematics problems which learners cannot do independently so that learners are given a chance to practice their ideas (Watson, 2009).

### **ANALYSIS OF THE TEXTBOOKS USING THE THEORIES ABOVE**

#### **Conceptual understanding**

As discussed earlier, Conceptual understanding refers to the grasp of mathematical ideas (Kilpatrick et al., 2001). In the textbooks analysed, Mathematics vocabulary is explained using multiple representations to communicate conceptual understanding of a word. In the extracts below, the definition of an arithmetic sequence is explained using symbols and words.



## Unit 1: Arithmetic sequences

In general we define an arithmetic sequence as follows:

$$a; a + d; a + 2d; a + 3d; a + 4d; a + 5d; \dots a + (n - 1)d$$

- $a$  is the value of the first term
- $d$  is the common difference between the terms,  $d = T_2 - T_1 = T_3 - T_2 = T_n - T_{n-1}$
- $T_n$  is the value of the term in position  $n$ , so  $T_n = a + (n - 1)d$
- $n$  is the position of a term and can only be a positive whole number, also known as a natural number.

**arithmetic sequence – a sequence of numbers with a common difference between consecutive terms**

Mathematics concepts are modelled using multiple representations to communicate conceptual understanding, for example, in the extract from one of the textbooks below, the textbook emphasises the use of three representations to solve given problems.

## Unit 2: Use Venn diagrams, tree diagrams and contingency tables to solve problems

When you have complex, wordy problems involving probabilities, it helps to visualise the data as diagrams. Diagrams make the information easier to understand. Using worked examples, we will discuss how to decide which type of diagram (Venn diagram, tree diagram or contingency table) would be most useful in each situation.

The textbooks which we analysed also use the strategy of concept mapping. The textbooks allow for development of a concept map to help learners discuss the important prerequisite learning concepts. Detailed Teaching guidelines on how the teacher may help learners to develop the concept map are given in the teacher's guide.

### Teaching guidelines

A line drawn parallel to one side of a triangle divides the other two sides proportionally.

To understand the theorem, the learners must understand and know that:

- the area of a triangle =  $\frac{1}{2}$  base  $\times$  perpendicular height.
- triangles which have the same base and same height are equal in area.
- triangles with different bases, but equal heights, are not equal in area.
- the ratio of the areas of triangles with equal heights is equal to ratio of the lengths of their bases.

If two triangles have a common vertex and their bases lie on the same straight line, then the ratio of their areas is equal to the ratio of their bases.

Extract from the Teacher's guide showing how to develop a concept map

### Procedural fluency

Procedural fluency refers to the skill in carrying out procedures flexibly, accurately and appropriately and this requires practice (Kilpatrick et al., 2001) and therefore variable examples must be given. The textbooks which we analysed provide students with various activities in order to practice mathematics. Some of the activities in the textbooks are: Various exercises in the learner book, Investigation, projects, mid-year exam papers (In the teacher's guide), term tests, control test book (separate book), question bank (on CD), exam practice papers in both learner book and Teacher's guide, and revision test topics. The textbooks also give learners examples with explanations of how to carry out the procedures appropriately as shown in the extract below.

So, taking the example given above and working in reverse order:

$$x^2 - 2x + y^2 + 10y = -17$$

$$x^2 - 2x + 1 + y^2 + 10y + 25 = -17 + 1 + 25$$

**REMEMBER**

The constant term of a perfect square trinomial, where the coefficient of the first term is 1, is always the square of  $\frac{1}{2}$  the coefficient of the middle term.

$$\left(\frac{1}{2} \text{ coefficient of } x\right)^2$$

$$= \left(\frac{1}{2} \times -2\right)^2 = 1$$

$$\left(\frac{1}{2} \text{ coefficient of } y\right)^2$$

$$= \left(\frac{1}{2} \times 10\right)^2 = 25$$

Add 1 and 25 to the right-hand side to balance the fact that those values were added to the left-hand side.

Now form both perfect squares, so

$$x^2 - 2x + 1 + y^2 + 10y + 25 = -17 + 1 + 25$$

same

$\sqrt{1}$

same

$\sqrt{25}$

sign

sign

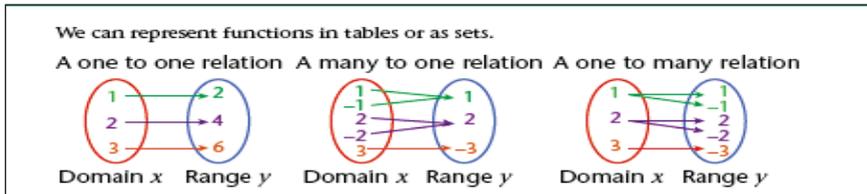
$$\therefore (x - 1)^2 + (y + 5)^2 = 9$$


### How the textbook uses Variation theory to develop procedural fluency

Marton et al. (2004) say that they are not advocating for an overall application of variation theory but they believe that variation theory can be used to enable learners to experience the features that are critical for particular learning and also to help learners in developing certain capabilities. There are four patterns of Variation: contrast, generalization, separation and fusion. These four patterns of variation are all catered for in the textbooks which we analysed as shown below.

#### Contrast

Contract refers to experiencing something that is not a critical feature of that object of learning so that they compare the different features (Marton et al., 2004). In the extract below, a function is represented in such a way that the learner is able to compare the different features such as domain and range of a function.



#### Generalization

This refers to giving various examples showing the critical features of the object of learning. In the extract below, the textbook gives many examples to capture different aspects.

**Determine the radius and centre of the following circles:**

**3.1**     $x^2 + y^2 = 121$

**3.2**     $3x^2 + 3y^2 - 9 = 0$

**3.3**     $(x - 1)^2 + (y - 5)^2 = 16$

**3.4**     $x^2 + (y + 15)^2 = 17$

**3.5**     $x^2 + 4x + y^2 - 8y = 5$

**3.6**     $3x^2 + 6x + 3y^2 - 18y + 12 = 0$

In the extract above, different aspects of the equation of a circle have been captured, for example, in 3.1, we see the aspect of both  $x^2$  and  $y^2$  having a coefficient of 1, in 3.2, the coefficient of the variables is not 1. From the activity above, the learners are able to practice different aspects but focusing on the equation of a circle.

### Separation

Separation refers to varying a particular aspect and keeping the other aspects invariant. This is also referred to as using controlled variation (Marton et al., 2004). In the extract above, separation has been catered for because the activities given are varying particular aspects of the equation of a circle such as the coefficients of the variables, the radius of the circle and the centre of the circle.

### Fusion

Fusion refers to discerning different critical features of the object of learning at the same time and this requires including more than one aspect in an example (Marton et al., 2004). In the extract below, different aspects are included in each of the questions, for example, in question 19, we see surds and binomials, in question 21, we see a fraction, a quadratic equation and binomial. All these questions are set in this way in order to allow the learner to practice different procedures.

$$19 \quad f'(x) \text{ if } f(x) = (\sqrt{x} - 3)(2\sqrt{x} + 3)$$

$$20 \quad f'(x) \text{ if } f(x) = 2x^5 - 3x^4 + \frac{1}{3}x^3 - 4$$

$$21 \quad g'(x) \text{ if } g(x) = \frac{3x^2 - 4x - 7}{x + 1}$$

$$22 \quad \frac{d}{dx}[(2x - 3)(4x^2 + 6x + 9)]$$

### Strategic competence

This refers to the ability to formulate, represent, and solve mathematical problems (Kilpatrick et al., 2001). Questions that require learner to come up with a strategy are given, for example, in the extract below, learners are expected to come up with a strategy to solve the problem.

- 1.3 The second and third terms of a geometric sequence are  $\frac{1}{3}$  and 1 respectively. Find the smallest value of  $n$  for which the sum of the first  $n$  terms exceeds 6.

### Adaptive reasoning

This refers to the capacity for logical thought, reflection, explanation, and justification (Kilpatrick et al., 2001). Questions that require learners to justify and give reasons for their answers are given in the textbooks, as shown in the extract below.



9.3.1 Calculate, with reasons, the size of each angle:

- a)  $\hat{E}$
- b)  $\hat{F}_2$
- c)  $\hat{B}_3$
- d)  $\hat{C}_1$
- e)  $\hat{C}_2$

9.3.2 Is CD a tangent to the smaller circle? Justify your answer.

9.3.3 Is EBFD a cyclic quadrilateral? Justify your answer.

### Productive Disposition

This refers to the ability to see mathematics as sensible, useful, and worthwhile. Productive disposition also refers to having belief in yourself and being able to see yourself as one who can do mathematics (Kilpatrick et al., 2001). The textbook provides techniques which teachers can use to help learners develop productive disposition, for example, the text books have sections on inclusive education and integration in the teacher's guide.

### Discussion and conclusion

The textbook used in the classroom is very important in determining how learners view mathematics (Mukucha, 2011). The textbook helps in enabling or restricting learners to communicate their mathematics ideas. The textbook should probe learners for higher mathematical reasoning by asking questions that require them to give more explanation and good justifications for their responses in the mathematics classroom. The textbook should also legitimate continuous assessment through exercises, tests, investigations and projects. This will enable learners to practice and hence reflect on their work (Aineamani, 2010). Marton et al. (2004) argue that the learners' focus should be on the direct object of learning, while the teacher's focus should be on both the direct object of learning and the indirect object of learning. Since the textbook is meant to act as the knowledgeable other as the teacher (Vygotsky, 1978), it has to also focus on both the direct and the indirect object of learning. Therefore, the textbook activities should be structured in such a way that the object of learning "comes to the fore of the learners' awareness" (Marton et al., 2004).

## REFERENCES

- Aineamani, B. (2010). Reasoning and Communicating Mathematically. In M. de villers (Ed.), *Proceedings of 18th meeting of SAARMSTE* Vol 1, University of Kwazulu-Natal: South Africa, pp 171-183.
- Dowling, P., (1996), A Sociological Analysis of School Mathematics Texts, *Educational Studies in Mathematics*, 31(4), 389-485
- Kilpatrick, J., Swafford, J. & Findell, B. (Eds). (2001). *Adding it up: Helping children learn mathematics*. Washington: National Academy Press. Pp 115-155.
- Marton, F., Runesson, U. and Tsui, B.M. (2004). The space of learning. In F. Marton & A. B. Tsui (Eds.), *Classroom discourse and the space of learning* (pp. 3 – 40), Mahwah, NJ: Lawrence Erlbaum.
- McKenzie, F. (2001). *Developing children's communication skills to aid mathematical understanding*, ACE papers (Student Edition), Issue 11.
- Morgan, C., (1995), An Analysis of the Discourse of Written Reports of Investigative Work in GCSE Mathematics, Unpublished, University of London
- Morgan, C., (2004), *Writing Mathematically: The Discourse of Investigation*, London: Falmer Press
- Muller, M. & Maher, C. (2009). Learning to reason in an informal Math After-School Program. *Mathematics Education Research Journal*, 21 (3), 109-119.
- Runesson, U. (2005). Beyond discourse and interaction. Variation: A critical aspect for teaching and learning mathematics. *Cambridge Journal of Education*, 35 (1), 69-87.
- Runesson, U. (2006). What is possible to learn? On variation as a necessary condition for learning. *Scandinavian Journal of Educational Research*. 50 (5), 397-410.
- Vygotsky, L. (1978). *Mind in Society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.
- Watson, A. (2009). *Paper 6: Algebraic reasoning. Key understandings in mathematics learning*. T. Nunes., P. Bryant and A. Watson, University of Oxford: Part 1: pp 1- 24. Nuffield Foundation: London.
- Watson, A. and Mason, J. (2006). Seeing an exercise as a single mathematical object: Using variation to structure sense-making. *Mathematical Thinking and Learning*. 8 (2), 91-111.



# EXPLORING THE USE OF ACTIVITY THEORY AS A FRAMEWORK FOR THE TEACHING AND LEARNING OF MATHEMATICS

Jayaluxmi Naidoo

University of KwaZulu-Natal: School of Education

*The purpose of this article was to explore the use of Activity Theory as a framework for the teaching and learning of mathematics. The final sample of participants comprised of six master teachers. Qualitative data was collected via master teacher questionnaires, observations of selected lessons, field notes of observations, individual interviews with master teachers and focus group interviews with learners. Data was analysed within an interpretive paradigm. Activity Theory was used as a theoretical lens within which to frame the study. The findings suggest that each master teacher modelled their classrooms as individual activity systems and the teaching and learning of mathematics was completed as a dynamic activity. These findings are important for advancing both teacher and curriculum development.*

## INTRODUCTION

Teachers are on a constant quest for the ideal strategy to advance the teaching and learning of mathematics. In South Africa, what makes this task more daunting is that all learners are required to select mathematics or mathematical literacy in grade 10 and continue with this choice until the end of their schooling career. Furthermore, learners are expected to write a common national examination at the end of their grade 12 year in mathematics or mathematical literacy regardless of glaring inequalities within school milieus across the country.

Although research focusing on pedagogic strategies within classrooms across various contexts has been conducted, there is currently a gap with respect to research focusing on the pedagogic strategies of teachers in South Africa. The question that arises is how do teachers teach for success in their classrooms? Hence the purpose of this paper was to explore the teaching and learning of mathematics within the framework of Activity Theory. Comprehensive lesson observations and video recordings provided important insights in strategies used by master teachers within differing social contexts. Master teachers in this study are expert teachers as identified by the KwaZulu-Natal Department of Education (KZN DoE). They are experienced teachers with the potential to mentor new teachers (Makapela, 2007). Identifying teaching strategies that support the effective and successful teaching of mathematics within differing social milieus could provide valuable insights for curriculum developers as well as teachers.

## THE TEACHING AND LEARNING OF MATHEMATICS IN SOUTH AFRICA

The advent of democracy in South Africa brought huge changes especially with respect to education. Education reformers in South Africa are concerned about the comparisons of South African learners with those of other nations, with respect to the apparent inability of South African youth to successfully participate in the mathematical global market place (Howie, 2003; Reddy, 2006).

Schools play a crucial role in preparing learners from different social backgrounds to meet the needs of an unequal society (Atweh, Bleicher, & Cooper, 1995).

Knowledge of mathematics is seen as an important asset for the progression of South African society. Similarly, Reddy (2005, p. 125) maintained that “mathematics and science are key areas of knowledge and competence for the development of an individual and the social and economic development of South Africa in a globalising world.”

With the introduction of mathematical literacy, it is a premise of this curriculum that it would prepare learners with the necessary mathematics skills required by individuals to function as critical, democratic citizens in modern society.

Mathematical literacy enables learners to become mathematically literate. To be mathematically literate suggests that learners are able to identify, understand and engage in mathematics as well as to make sound judgements about the role that mathematics plays in the real world (Kotze & Strauss, 2006).

### THE USE OF VISUAL TOOLS IN SCHOOLS

Visualisation essentially means the ability to form and negotiate a mental image necessary for problem-solving in mathematics. Visual images refer to the representation of the visual appearance of an object, e.g. its shape, colour and size (Van Garderen, 2006). They also play a decisive role in promoting critical thinking (Rezabek, 2008), learning and communication in the mathematics classroom. This is so because visualisation encourages the use of the concrete to conceptualise abstract concepts and ideas (McLoughlin, 1997; Presmeg, 1997; Solano & Presmeg, 1995). Whilst mathematics encompasses many abstract notions, with visualisation, and visual tool use, these abstract notions are made more accessible to the learner.

Teachers often use visual tools unknowingly in class, for example, when they resort to the use of gestures, graphs, shapes, lines and diagrams. A gesture is any physical body movement (Maschietto & Bartolini Bussi, 2009) that assists in a communication function (Sfard, 2009). Learning environments that incorporate visual tools and technology add value to lessons. In these learning environments, learners are able to interact easily with concepts that were once considered abstract.



For example, teachers teaching transformations in geometry may use technology to manoeuvre rotations and reflect images. Learners are then able to see these transformations, allowing them to concretise these once abstract mathematics concepts. Technological tools like the smart board and the calculator enable teachers and learners to display ideas and allow for multiple interpretations. These interpretations may be discussed, interpreted and revised based on feedback from peers (McCoughlin, 1997).

Additionally in mathematics there are various computer-based packages (e.g. *Geometer's Sketchpad*; *Cabri Geometry*; *GeoGebra*; *GeoProof*; *Cinderella* and *Graphmatica*. etcetera) available for the teaching and learning of mathematics. By using software programmes like *Geometer's Sketchpad*, the teacher frees up more time to interact with the learners and to ask conceptual, probing questions (Steer, de Vila & Eaton, 2009). This is ideal for occasions when the learner cannot see or understand what the teacher is talking about within a mathematical context. In addition, these tools have the added benefit of allowing learners to discover rules and generalisations for themselves (ibid, 2009).

## THEORETICAL FRAMEWORK

Activity plays an important role in mathematics learning and development (Grives & Dale, 2004), however, very little activity of significance is accomplished individually (Jonassen & Rohrer-Murphy, 1999). Activity theory was used as a framework for the study to account for the systems that link mathematics, learning and the differing social milieus. Based on the principles of activity theory, activity and learning are interactive and interdependent (Jonassen, 2002). This theory is based on the assumption that all human actions are mediated by tools and cannot be separated from the social milieu in which action is carried out. Activity theory in this study provides the framework for describing the structure, development and context for the activities that were supported through the use of visual tools. These tools differed based on the context within which each school was located. For example, during the teaching of transformation geometry, one master teacher used the smart board; another master teacher used a stick with different coloured elastic bands and the next master teacher used paper folding and gestures. These diverse tools were used to teach the concept of rotation and reflection within transformation geometry.

In this study, the activity system under the microscope is the act of teaching and learning in mathematics classrooms. The community within the activity system refers to a group of individuals who share a common objective. Hence, learning is not seen as an isolated act. Learning occurs as individuals interact with each other, and these interactions are mediated by tools. Barab, Schatz and Scheckler (2004) proposed that activity theory emphasised the mutual nature of learning and doing, of tool use and community and of content and context. To clarify, as the learning community within each activity system in the study works and solves problems together, they develop a new set of values and notions.

These values and notions may not be appreciated or understood within other communities and in other contexts. Engeström's (1987, 2001), second generation activity theory model was used in this study. To situate the activity theory model within the context of this study, the subject is defined as the master teacher, the instruments are the visual tools that are used to teach mathematics and the object is the development of the mathematics content. The communities in this study are the learners within the mathematics classroom, the teachers, the staff at each school and the parents within the community. The subject belongs to a community that is governed and mediated by rules and division of labour. Essentially members of the community collaborate with each other to achieve the outcome of the activity system. In this study the outcome of each activity system was the teaching and learning of mathematics. The model that follows (Figure 1) emerged from this study. This model graphically demonstrates how activity theory was used in the teaching and learning of mathematics. The smaller external activity system that is illustrated in the figure varies for each master teacher.



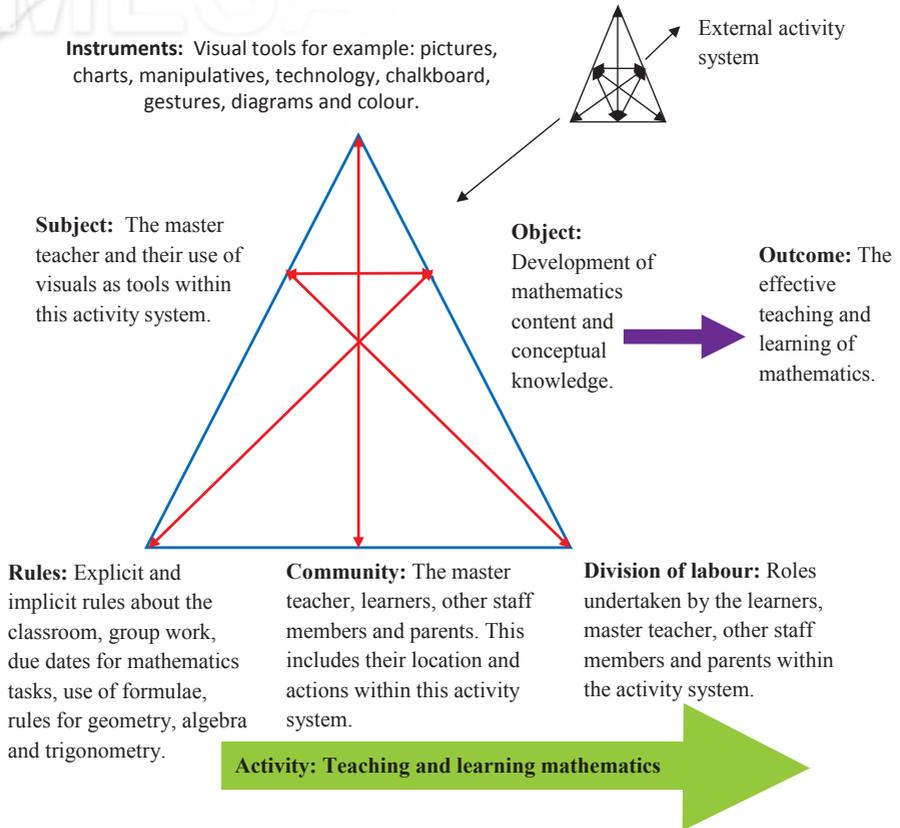


Figure 1: Conceptual model of the human activity system within this study adapted from Engeström (1987, p. 78)

## RESEARCH METHODOLOGY

Gatekeeper access was obtained from the KZN DoE first before forty five schools were invited to participate in the study. The schools were selected based on convenience and accessibility for the researcher. Twenty out of 45 schools responded positively to the invite. Ten schools were selected at random to be a part of the pilot study and the remaining ten schools participated in the final study. Based on data collection, analysis and intensive coding, the final sample comprised of six master teachers teaching at six different schools located in KZN. The final sample is depicted in Table 1 that follows.

Number	Master Teacher	School
1.	Alan	Orchard Secondary
2.	Dean	Daisy Secondary
3.	Karyn	Rose Secondary
4.	Maggie	Lily Secondary
5.	Penny	Tulip Secondary
6.	Sam	Carnation Secondary

Table 1<sup>1</sup>: Participants in the final sample (Adapted from Naidoo, 2012, p. 3)

Learners were selected for focus group interviews at each of the six schools. These learners were purposively selected based on the level of interaction with their teacher and peers in the classroom. The focus group thus comprised of learners who interacted frequently, average and not at all.

### Data collection

Prior to the pilot testing of the questionnaire, items on the questionnaire were discussed with colleagues within similar research areas. After minor edits, the questionnaire was pilot tested. Ten of the twenty schools were selected at random to participate in the pilot study. In stage 2 a questionnaire was sent to each of the remaining ten schools. The master teacher questionnaire was a pen and paper questionnaire whereby the master teachers were asked various questions relating to their teaching strategies and resources used. The questionnaire was analysed and coded in preparation for the next stage of data collection. Stage 3 involved the observation and video recording of at least three Grade 11 mathematics lessons that were taught by each of the six master teachers. After analysing and coding the observations, Stage 4 - the master teacher interview phase commenced.

---

<sup>1</sup> Pseudonyms are used to protect the identity of the schools and teachers.



Each of the six master teachers selected were interviewed using a semi structured interview schedule. Each interview was recorded. Video clips were provided for the master teachers to view. The video clips were selected at critical moments in each of the master teacher's lesson. These critical moments focused on the master teacher's use of visuals as tools in the lesson. The master teachers were asked questions pertaining to their use of the visuals as tools. The master teachers' responses were probed were necessary to ensure that there were no misinterpretations or misunderstandings. After analysing the master teacher interviews, focus group interviews were conducted with learners from each of the six schools. A semi structured focus group interview schedule was used and each focus group interview was recorded. Learners were provided with the opportunity to view the same video clips that were shown to their teachers. Learners were asked questions about the use of these visuals as tools in their classrooms.

## **FINDINGS AND DISCUSSION**

### **The activity systems**

Based on the lesson observations and interviews it was evident that each of the classrooms functioned as an activity system. Each member of the classroom community served a specific purpose and all role players worked in collaboration with the master teacher (subject) to achieve the outcome of the activity. In order for this outcome to be achieved rules were followed and specific instruments were used. The analysis of the six activity systems highlighted different external activity systems that influenced the teaching and learning of mathematics. These external activity systems are discussed below.

### **The history of poor resourcing:**

For decades the history of poor resourcing created disadvantages across schools in South Africa. Sharing of resources causes conflicts with the planning and time management of lessons. For effective teaching and learning to occur it is necessary to make maximum use of teaching time (Pollard & Triggs, 1997). Apart from the sharing of material resources, human resources were also limited. This implied that teachers at these schools spent a great deal of their non-teaching periods serving relief for teachers that were not present at the school. This lack of an adequate support structure, limits the teacher's time for effective planning and preparation for future lessons.

Limited human resources impact negatively on both the community and division of labour within the activity system. Members of the teaching staff within the learning community were burdened by the added pressure of providing adequate support to all learners within the learning community. Teachers were compelled to take on more responsibility and a greater workload. This external activity system led to teachers being overworked, worn-out and stressed in the classroom. This could lead to more teachers being absent than at other advantaged schools.

Moreover, as a result of the limited resources, the master teacher was required to be more resourceful than colleagues at other more advantaged schools. In many cases, the teacher would need to reflect on previous experiences and use a combination of tacit and explicit knowledge in order to be effective in the classroom. For example in Carnation Secondary, the lack of resources caused tension between the visual mediating tools and the master teacher (Sam). Sam reflected on his practice and used manipulatives and visual tools that were easily available to learners within the classroom environment. He used bricks, desks, coloured chalk and mental images to assist his learners. The use of these visual tools influenced the use of instruction time in the classroom. More time was spent in drawing diagrams and mediating tools than in the interaction and engagement within the classroom.

### **The privileges of a well-resourced school**

This activity system at Rose and Lily Secondary School were privileged in all respects when compared to the other activity systems in this study, because these schools had better material and human resources. Teachers at these schools were in abundance, and they were well qualified. This external activity system influenced all aspects of this activity system. The effects on the rules within these activity systems were minimal. Most learners at these schools came from privileged backgrounds and other advantaged primary schools. They were accustomed to the rules of this type of learning environment. The minority of the learners who did not come from privileged backgrounds conformed to what was socially acceptable within the learning environment. They developed a shared understanding with other members of the learning community; this suggested that these learners assumed the dominant practices of this activity system (Naidoo, 2006). The teachers at these schools came from privileged backgrounds and could identify with the learners within these activity systems. The teachers had a range of tools from which to choose and they used their tacit knowledge to make tool use within their activity system beneficial. This dynamic use of tools led to a varied approach to the teaching and learning of mathematics.

With respect to issues of community and division of labour, most of the learners came from better socio-economic backgrounds and hence had the parental support ensuring success at schools. Most parents could afford to provide their children with extra resources to assist with effective learning. These resources included, for example, study guides, computer access, internet access and extra tuition. This extra parental support influenced the roles of the learning community and the division of labour within the classroom. Learners were able to take more responsibility for their own learning. Additionally, being well resourced schools, there was more funds available for employing additional staff members as the need arose.



This relieved the burden of relief teaching and support. Responsibilities could be shared with other staff members and hence more time could be spent on planning and preparing for lessons. What is evident from this discussion is that the external activity system of privilege positively influenced the object and eventual outcome of both Karyn's and Maggie's activity system.

### **The mediating tool of language**

The mediating tool of language affected how the rules and instructions were understood within the learning environment. For example at Daisy Secondary, learners came from different language backgrounds and at times these differences created issues of miscommunication within the classroom. The majority of learners at the school spoke English as a second or even third language. Due to differences between the dominant language at home and the language of instruction at school, certain mathematics concepts and rules had to be revisited. This revisiting of mathematics concepts added pressure on the time allocated for syllabus coverage. Regardless of these external conflicts, the syllabus had to be completed in time for the national tests and examinations. To overcome these conflicts, the master teacher (Dean) had to be reflective and resourceful in the classroom.

Within Penny's activity system, language influenced how she taught the lessons as well as how rules and instructions were interpreted within the classroom context. Language is a cultural matter; it is a way of communicating meanings and of coding events. Generally, children attain a basic mastery of their mother tongue before they start school. Since learners find it difficult to follow instructions in a language that is not their mother tongue, this may account for their poor academic performance (Mwamwenda, 2004). This caused additional conflict and tension within Penny's activity system. Likewise Zevenbergen (2001a) proposed that the language that learners learn from their homes is prone to locate them more or less favourably at school. This notion is dependent on the correlation between the home and school languages. This suggests that the home and language are connected, with the home being very significant to how the learner communicates within the classroom (Naidoo, 2006).

The links between the school and the home tend to be stronger with families from advantaged backgrounds than with families from disadvantaged backgrounds. Learners from advantaged backgrounds use an "elaborated code" whilst learners from disadvantaged backgrounds use a "restricted code" with respect to language (Bernstein, 1971, p. 76). Learners from advantaged backgrounds have access to both codes and this locates them as having a more dependable voice within the community. Learners from disadvantaged backgrounds may not have access to the elaborated code and this partly explains their underachievement at school (Boaler, 2002).

The personification of characteristics of tastes, disposition and language can be seen to be the structure of habitus. “Habitus” according to Bourdieu is the embodiment of culture and presents the lens through which the world is construed (Zevenbergen, 2001b, p. 202). Each activity system in this study exhibited their own ‘habitus’ whereby each learning community constructed the learning of mathematics diversely. Each learning community in this study used their own symbols and visual tools to make the mathematics more accessible to members within the learning community. Within this activity system, Penny used many differing strategies to compensate for the differing backgrounds of her learners. Penny also used a combination of group work and individual work during her lessons.

Within Sam’s activity system the language used at home caused conflict with the language of instruction. This conflict posed problems for Sam when he tried to engage learners with the mathematics being taught. If one does not have exposure or experience of certain words and concepts, one would not be able to communicate effectively within a classroom context. One would be functioning within Bernstein’s “restricted code” of language (Bernstein, 1971, p. 76). Rules related to the content taught and concepts learnt will most likely be misunderstood. Thus, apart from the use of visual images to assist in making the language of mathematics more comprehensible and accessible, Sam encountered an added dilemma. Sam needed to ensure that the visual tools used would be understandable to all his learners. He needed to ensure a levelling of the mathematics ‘playing field’ before any teaching and learning could occur.

### **The lack of discipline**

The external activity system of lack of discipline affected the rules of the classroom, the learning community and the division of labour within Penny’s activity system. In every classroom there are rules to be followed. This assists in attaining Levels 2 to 4 of Maslow’s hierarchy of basic needs (Figure 2). Maslow’s hierarchy of needs provides teachers with an illustrated version of basic needs that ought to be met for successful teaching and learning to occur. The diagram that follows depicts Maslow’s hierarchy of basic needs.

<b>KEY TO MASLOW’S HIERACHY OF NEEDS</b>
5. Self-actualisation needs
4. Esteem needs
3. Belongingness and love needs
2. Safety needs
1. Physiological needs

Figure 2: Maslow’s hierarchy of needs. Adapted from Pollard and Triggs (1997, p. 201)



Maslow's hierarchy of needs provides teachers with an illustrated version of basic needs that ought to be met for successful teaching and learning to occur. Due to the lack of discipline at some schools, rules are compromised and more responsibility is shifted to the learning community which in turn leads to an increase in the division of labour.

In Tulip Secondary the master teacher (Penny) used various strategies to ensure that the learners' behavioural problems that were evident in other classrooms did not manifest itself in her classroom. Whilst Penny used her knowledge gained from her vast experience to make her classroom more conducive to teaching and learning for her learners, she would not necessarily have used these strategies and methodologies if this external activity did not begin to manifest itself in her classroom. The reason Penny did not have a discipline problem in her classroom, as experienced by the other teachers at the school, was that she used diverse teaching strategies and methodologies in her classroom. This included the use of scaffolding, stories, pictures and colour.

The lack of discipline also exhibited itself in Sam's classroom at Carnation Secondary. Whilst this did not hamper Sam in his teaching this was significant. This suggested that Sam could control how the learners behaved within his activity system. He did this by making his classroom milieu more accessible, inviting and conducive to the learning of mathematics. He used pictures, diagrams, concrete manipulative and gestures to attain this atmosphere.

### **National examinations and testing**

An additional external activity system that influenced the activity system within Sam's classroom was that of the national examinations and tests. Regardless of the context of the school, all schools write national examinations and tests. When teaching time is spent on drawing diagrams and levelling out the mathematics 'playing field', this has an impact on syllabus coverage. The added pressure on the teacher of completing the syllabus in a shorter period affects the community within the activity system. The community becomes overworked and the division of labour is unfairly distributed. To overcome this issue Alan and Dean spent many afternoons and weekends at school providing extra classes for their learners in order to complete syllabus and provide support to their learners.

### **Parental involvement**

The sixth theme that was evident was the lack of parental involvement at some schools. For example, the learners who attended Carnation Secondary came from less privileged backgrounds. Whilst the parents wanted their children to succeed in school and obtain the best possible education, the home environments caused conflict with the community and the division of labour within this activity system. Not all parents could provide educational support for their children. Most parents could not afford additional materials and extra tuition to assist in improving their children's educational abilities. This added pressure on the community within the school to

provide an environment that supported these needs. This in turn added extra responsibilities on the already overextended teachers at the school. This caused an unfair disadvantage when compared to other schools within the more affluent suburbs of KZN. The division of labour amongst the learning community was unfairly distributed.

## CONCLUSION

What came out strongly in the interviews and observations was that whilst the teachers had their own techniques of teaching, their methods were both grounded in tradition and modified as they taught; each master teacher was willing to share their practice. It was through this practice that they could comment on the usefulness and appropriateness of their visual tools for specific sections in mathematics. This resonates strongly with the notions of activity theory, whereby human beings mediate their activities using tools. The results of the study exhibit that the use of visuals as tools within the different activity systems made the abstract nature of certain mathematics concepts more concrete, these tools made certain mathematics concepts easier to remember, and in general the mathematics lessons became more fun and interesting. The high level of student engagement and interaction in the different activity systems seems to indicate that through the use of activity theory the teaching and learning of mathematics was more effective.

A recommendation of the study is that mathematics teachers employ the use of visuals as tools in their classrooms through the use of activity theory as a framework for their lessons. This would make lessons more successful and teachers would achieve their aim of promoting the effective teaching and learning of mathematics.

## REFERENCES:

- Atweh, B., Bleicher, R. E., & Cooper, T. (1995). Social context in mathematics classrooms: Social critical and sociolinguistic perspectives. Paper presented at the annual meeting of the American Educational Research Association, San Francisco.
- Barab, S., Schatz, S., & Scheckler, R. (2004). Using activity theory to conceptualize online community and using online community to conceptualize activity theory. *Mind, Culture and Activity*, 11(1), 25 - 47.
- Bernstein, B. (1971). *Class, codes and control*. London: Routledge and Kegan Paul, Publishers.
- Boaler, J. (2002). Learning from teaching: exploring the relationship between reform, curriculum and equity. *Journal for Research in Mathematics Education*, 33(4), 239 - 258.
- Engeström, Y. (1987). *Learning by expanding: An activity-theoretical approach to developmental research*. Helsinki: Orienta-Konsult.
- Engeström, Y. (2001). Expansive learning at work: toward an activity theory reconceptualization. *Journal of Education and Work*, 14(1), 134 - 156.
- Groves, S., & Dale, J. (2004). Using activity theory in researching young children's use of calculators. Paper presented at the Annual conference of the Australian Association for Research in Education, Melbourne.
- Howie, S. J. (2003). Language and other background factors affecting secondary pupil's performance in mathematics in South Africa. *African Journal of Research in SMT Education*, 7, 1 - 20.
- Jonassen, D. H. (2002). Learning as activity. *Educational technology*, 42(2), 45-51.



- Jonassen, D. H., & Rohrer-Murphy, L. (1999). Activity theory as a framework for designing constructivist learning environments. *Educational Technology Research and Development*, 47(1), 61 - 79.
- Kotze, G. S., & Strauss, J. P. (2006). Contextual factors of the mathematics learning environments of grade 6 learners in South Africa. *Pythagoras*, 63, 38 - 45.
- Makapela, L. (2007). Dinaledi schools making progress. Retrieved 13th October 2008, from South Africa. info. Gateway to the nation <http://www.southafrica.info/about/education/dinaledi-161007.htm>
- Maschietto, M., & Bartolini Bussi, M. G. (2009). Working with artefacts: gestures, drawings and speech in the construction of the mathematical meaning of the visual pyramid. *Educational Studies in Mathematics*, 70, 143 - 157.
- McLoughlin, C. (1997). Visual thinking and telepedagogy. Paper presented at the ASCILITE, Perth, Australia.
- Mwamwenda, T. S. (2004). Educational psychology. An African perspective. Sandton: Heinemann Higher and Further Education.
- Naidoo, J. (2006). The effect of social class on visualisation in geometry in two KwaZulu-Natal schools, South Africa. Education. University of Nottingham. Nottingham.
- Naidoo, J. (2012). Teacher reflection: The use of visual tools in mathematics classrooms. *Pythagoras*, 33(1), 1 - 9.
- Pollard, A., & Triggs, P. (1997). Reflective teaching in secondary education. London: Cassell.
- Presmeg, N. C. (1997). Generalization using imagery in mathematics. In L. D. English (Ed.), *Mathematical reasoning. Analogies, metaphors and images* (pp. 299 - 312). Mahwah, New Jersey: Lawrence Erlbaum Associates, Publishers.
- Reddy, V. (2005). State of mathematics and science education: Schools are not equal. *Perspectives in Education*, 23(3), 125 - 138.
- Reddy, V. (2006). The state of mathematics and science education: schools are not equal. In S. Buhlungu, J. Daniel, R. Southall & J. Lutchman (Eds.), *State of the Nation. South Africa 2005 - 2006*. (pp. 392 - 416). Cape Town: HSRC, Publishers.
- Rezabek, L. L. (2008). Why visual literacy: Consciousness and convention. *TechTrends*, 49(3), 19 - 20.
- Sfard, A. (2009). What's all the fuss about gestures? A commentary. *Educational Studies in Mathematics*, 70, 191 - 200.
- Solano, A., & Presmeg, N. C. (1995). Visualization as a relation of images. Paper presented at the 19th Conference of the International Group for the Psychology of Mathematics Education, Brazil.
- Steer, J., de Vila, M. A., & Eaton, J. (2009). Trigonometry with year 8: Part 1. *Mathematics Teaching*, 214, 42 - 44.
- Van Garderen, D. (2006). Spatial visualization, visual imagery, and mathematical problem solving of students with varying abilities. *Journal of Learning Disabilities*, 39(6), 496 - 506.
- Zevenbergen, R. (2001a). Language, social class and underachievement in school mathematics. In P. Gates (Ed.), *Issues in mathematics teaching*. (pp. 38 - 50). London: Routledge Falmer Publishers.
- Zevenbergen, R. (2001b). Mathematics, social class and linguistic capital: An analysis of mathematics classroom interactions. In B. Atweh, H. Forgasz & B. Nebres (Eds.), *Sociocultural research on mathematics education* (pp. 201 - 215). Mahwah, New Jersey: Lawrence Erlbaum Associates, Publishers.

# CONSTRUCTION ROUTES IN THE SOLUTION OF COMPASS AND STRAIGHTEDGE CONSTRUCTION PROBLEMS: GEOMETRIC REASONING OF FIRST YEAR MATHEMATICS EDUCATION STUDENTS

Bharati Parshotam and Erna Lampen

Wits School of Education

*In this paper we share our initial analysis of the construction routes of a sample of 23 first year mathematics education students who solved a compass and straightedge construction problem as an assessment task. Four construction routes were identified which provide a window on their geometric reasoning about the rhombus they had to construct. We describe the apparent theoretical geometric reasoning and spatio-graphic reasoning of each construction route and reflect on possible reasons for unsuccessful constructions. These initial findings are important for teachers and teacher educators who want to use constructions as a tool for the development of geometric reasoning.*

## INTRODUCTION

Mathematics teacher education in South Africa, as in other developing countries has to take note of tendencies in developed countries to shift towards the use of dynamic computer based learning in schools. Yet, in South Africa the reality is that the majority of schools do not have access to the necessary technology and the majority of teachers do not have the knowledge to use computer based artifacts for teaching. With the eye on the future mathematics teacher education nevertheless has to provide access to such technology and develop the reasoning to accompany learning and teaching with technology. At the same time it has to provide alternative non-technological tools for teaching which can provide access to computer age reasoning. We argue that the compass and straightedge is such a tool.

The South African school mathematics curriculum includes compass and straightedge constructions in Grades 7 to 9, but judging from examples in the CAPS document actual construction tasks tend to be procedural and limited to a list of basic constructions, such as the construction of a line perpendicular to another, halving a line segment and duplicating and halving angles. Evident from textbooks, the basic constructions are taught as an end in themselves and teachers are not likely to utilize them in geometric problem-solving or to develop geometric reasoning. In this paper we analyse the construction routes of a sample of 23 first year mathematics education students who constructed a rhombus from limited theoretical information. The construction routes provide information about their geometric reasoning.



## CONSTRUCTIONS AND PROOF REASONING

Martin (1998, p. 2) explains the relationship between theorems and proofs and constructions as follows:

“In general, a *theorem* is a statement that has a proof based on a give set of postulates and previously proved theorems. A *proof* is a convincing argument. A problem in Euclid asks that some new geometric entity be created from a given set. We call a solution to such a problem a *construction*. This construction is in itself a theorem, requiring a proof and having the form of a recipe: If you do this, this, and this, then you will get that. Such a mathematical recipe is called an algorithm. So a construction is a special type of theorem that is also an algorithm. (We hesitatingly offer the analogy: *Problem*: Make a pudding; *Construction*: Recipe; *Proof*: Eating)”

Martin’s baking analogy suggests that the proof is only the spatio-graphical, perceptual object that is the end product of the construction. Laborde (2005, p. 174) is more explicit about the demands posed by construction problems, namely to produce mappings between the theoretical and spatio-graphic domains. With a compass and straightedge construction the spatio-graphic properties provide the visual verification of the correctness of the constructed figure, while the theoretical properties guide the construction process. In the construction problems we posed in our first year geometry course for mathematics education students, an algorithm has to be developed based on the theoretical-geometric properties of the end product and the affordances and constraints of the compass and straightedge as tools. The algorithm and the spatio-graphic object together serve to provide the convincing argument. According to Laborde (2005, p. 160) a key aspect of geometry learning and reasoning is to distinguish between incidental and necessary spatio-graphical properties of geometric objects. This is exactly the opportunity afforded by Euclidean constructions. Whether done with paper, pencil and compass and straightedge, or with dynamic tools, Euclidean constructions are based solely on distance-relationships between points, and reasoning has to start with the construction of a basic length or distance (the radius of the initial circle). All points that are used in the subsequent algorithm have to be determined as intersections of arcs, lines or segments, or endpoints of segments. Points that are not created by such intersections or segmentations have arbitrary or incidental position. In South-African classrooms learners are very seldom required to draw or construct the geometrical object of a problem and textbooks provide diagrams for all problems, with the result that learners often find it difficult to discern the necessary or incidental properties of points and segments. Our observations about students’ strategies to solve riders is that they tend to fill in on a given diagram all immediately available measurements or congruencies and then hope for a solution to “jump in the eye”. They are hard pressed to reason about the primacy relationships between parts of the figure. We concur with Laborde (2005) that it is detrimental to geometric reasoning when theoretical properties are stressed almost to the exclusion of spatio-graphical properties in teaching.

## CONSTRUCTIONS AND THE VAN HIELE LEVELS OF GEOMETRIC REASONING

The Van Hiele (1959) theory of geometric reasoning holds that such reasoning develops from gestalt-like visual reasoning, described as recognition at Level 1, to reasoning based on the analysis of the properties of geometric objects at Level 2, and further to reasoning with the ordered relationships between properties of objects at Level 3. At Level 4 learners are able to develop longer sequences of deduction and begin to understand theorems and proofs. The reasoning levels are hierarchical and language or verbal reasoning plays a significant role in the development through the levels. Van Hiele stressed the importance of active tactile involvement of learners in order to develop their reasoning from visual to analytic. In the same vein De Villiers claims that the transition from Level 1 to Level 2 “involves a transition from enactive-iconic handling of concepts to a more symbolic one.” (De Villiers, 2010, p. 2)

While we were aware that our students were unlikely to be at Level 3, since geometry was not compulsory in their high school curriculum, we aimed to challenge our students to reason with the logical relationships between properties of figures. Such reasoning is described as Van Hiele Level 3 reasoning and indicated by the objects of their reasoning such as “noticing and formulating logical relationships between properties, for example that equal opposite sides implies that the sides are parallel.” (De Villiers, 2010, p. 3) Until dynamic geometry programs became available research about Van Hiele thought levels was mostly based on the verbal contributions of participants. For example, Burger and Shaughnessy’s (1986) descriptors of Van Hiele level reasoning is given exclusively in terms what participants say, and makes no mention of actions and their implications for geometric reasoning. Van Hiele research conducted in contexts of geometry software describes the dragging and construction actions of participants as they investigate and solve problems. The body of evidence is growing that the opportunities for goal directed action on geometric objects is an important aspect in the development of geometric reasoning. (De Villiers, 2010; Idris, 2009) Yet, we are not aware of research that analyse geometric reasoning during compass and straightedge construction in terms of Van Hiele levels. Our analysis of the task demands of problem-solving by construction is that at least Level 3 reasoning is required.



## VISUAL AND THEORETICAL ASPECTS OF CONSTRUCTIONS

In the context of goal-directed action on dynamic geometric objects, the issues of the role of visualisation and the influence of the spatio-graphic diagram in relation to reasoning are given new priority. Early proponents of the importance of spatio-graphic properties like Fischbein (1993) and Mariotti (1995) describe geometric reasoning as conceptual-figural, and highlight pervasive conflicts between the two modes of reasoning. More recently Laborde (2005) reformulated the theoretical and visual aspects of geometric reasoning in terms of the referents rather than mental actions. Fischbein and Mariotti refer to geometric reasoning in non-dynamic contexts, while Laborde takes her thesis from research in dynamic contexts. Laborde (2005, p. 161) distinguishes between two domains, the *theoretical domain* (T) comprising of geometrical objects and relations; and the *spatio-graphic domain* (SG) – that of diagrams on paper or on the computer screen, or, importantly “movement produced by a linkage point of a machine.” The theoretical and spatio-graphic domains are not independent and geometric reasoning requires repeated moves between the domains. According to Laborde geometry teaching should aim at integrating the T and SG domains. We argue that a pair of compasses is a machine with a linkage point, and as such included in the spatio-graphic domain, while constructions with these tools require reference to geometric relationships between properties of the object to be constructed.

As predicted by Van Hiele theory, the visual aspects of constructions are most influential for novice constructors. In the one study we could find of reasoning during compass constructions, Tapan and Arslan (2009) analysed pre-service teachers’ compass constructions and accompanying justifications and concluded that visual and naïve empirical reasoning were the norm. The participants in their study struggled to organize constructions into theoretically sound algorithms.

### THEORETICAL FRAMEWORK

We framed our research according to Laborde’s (2005) distinction between the use of theoretical and spatio-graphic properties in geometric constructions - in particular, her thesis that the meaning of a geometrical activity resides in the weights of these properties as they interplay during construction. Analysis of the referents in a construction algorithm points to whether a student assigns theoretical or spatio-graphic meaning to the task. In other words, whether a construction is judged as successful based on mainly theoretical or visually observable spatio-graphic properties. This framework is an initial attempt to gain information about students’ geometric reasoning from their constructions. We analyse the theoretical and spatio-graphic properties used in each construction route, and infer the meaning assigned to the construction. In our analysis the meaning emerges from an analysis of the properties that fully determine the outcome of the construction.

For example, if the construction marks indicate that side lengths were measured and constructed when they would have been determined by the intersection of rays, we infer that the spatio-graphic referent of equal side lengths guided the student, rather than the theoretical sufficiency of the preceding steps in the construction algorithm.

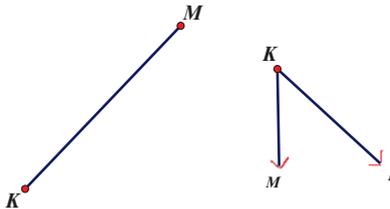
### RESEARCH QUESTION AND METHOD

We aim to get initial answers the following questions:

- 1) What construction routes or algorithms are used when first year mathematics education students solve a construction problem?
- 2) What are the theoretical and spatio-graphic reasoning evident from the construction routes?
- 3) What meaning (theoretical or spatio-graphic) is assigned to the construction as evident from the concluding steps in the construction?

### The construction problem

Construct a rhombus.  $KM$  is a diagonal of the rhombus and  $MKL$  is an angle. Use only compasses and a straightedge.



The students' constructions were categorized as successful or unsuccessful; and each category was further categorized in terms of the construction route. This allowed us to compare inferred reasoning processes and to develop hypotheses about factors that constrained the solution of the problem.

The construction problem draws on reasoning with the properties of a rhombus that are dependent on the diagonal. In order to solve the problem, students have to image the end product – a rhombus with diagonal  $MK$  and an angle  $MKL$ . The position of angle  $MKL$  in the rhombus requires an interpretation of the labels and knowledge of the conventions of labeling. The leg  $KM$  of the angle is also the diagonal  $KM$ , but while the length of the diagonal is determined, the lengths of the legs of the angle are not determined. Once the labeling conventions are sorted out, several solution routes are available based on the properties of the rhombus that are dependent on the diagonal.



## RESULTS

Four major successful construction routes were identified.

### Construction route 1 (n=2)

Construct KM. At K and M construct angles congruent to angle MKL to form isosceles triangle KML. Complete the rhombus by constructing KN and MN congruent to KL.

*Spatio-graphic reasoning:* A rhombus has four congruent sides; a diagonal of a rhombus bisects the rhombus into two congruent isosceles triangles.

*Theoretical reasoning:* The given angle on the diagonal is one of the base angles of an isosceles triangle on the diagonal. The size of the given angle and the length of the diagonal determine the length of the sides of the rhombus.

*Meaning of the construction:* Spatio-graphic: Find the length of the sides of the rhombus and construct all four sides. (The final step is the measurement and the construction of the remaining two isosceles sides.)

### Construction route 2 (n=8)

Construct line segment KM. At K, construct angle LKN equal to double angle MKL. At M, construct angle LMN equal to double angle MKL. The intersections of the arms of angle LKN and angle LMN completes the rhombus.

*Spatio-graphic reasoning:* The diagonals of a rhombus are symmetry lines. (The second diagonal is often drawn but no properties of the intersection of the diagonals are indicated with markings or tested by construction).

*Theoretical reasoning:* If the given diagonal of a rhombus is a symmetry line, then the angles at the same vertex on either side of the diagonal are congruent. With this construction route the rhombus is fully determined once angles LKN and LMN are constructed.

*Meaning of the construction:* Theoretical: Congruent sides are the result of equal opposite angles bisected by a diagonal.

### Construction route 3 (n= 1)

Construct Segment KM and its perpendicular bisector. At K construct angle MKL congruent to the given angle. The intersection of leg KL and the perpendicular bisector of KM defines the side length of the rhombus. Connect M with L. Complete the rhombus by constructing sides KN and MN congruent to KL. (A variation on this route is to construct two adjacent angles, each congruent to MKL, at K and hence obtain two intersections with the perpendicular bisector of KM).

*Theoretical reasoning:* If the second diagonal bisects the given diagonal perpendicularly then the given angle determines the position of the vertex formed by the intersection of the second diagonal and the leg of the angle which is not the given diagonal. The rhombus is fully determined when the right triangle between the diagonals is constructed.

*Spatio-graphic reasoning:* The second diagonal is bisected (perpendicularly) by the given diagonal. Markings indicate the congruent sections of the second diagonal, but the congruence is not constructed or tested. Opposite sides are marked parallel purely on visual grounds. Parallel properties are not verified.

*Meaning of the construction:* Spatio-graphic: Find the length of a side of the rhombus and construct all four sides. (The bisection of the second diagonal is not used to construct the fourth vertex, but the constructed side length is copied and constructed explicitly three more times.)

#### **Construction route 4 (n=6)**

Construct line segment KM and its perpendicular bisector. At K construct angle MKL congruent to the given angle, and at M construct the alternate angle KMN congruent to the given angle. The intersections of the angles with the perpendicular bisector of KM define the side lengths of the rhombus. Complete the rhombus connecting L with M and K with N.

*Theoretical reasoning:* At least one diagonal of a rhombus bisects the other diagonal perpendicularly; opposite sides of a rhombus are parallel, alternate angles formed by the diagonal and the sides of a rhombus are congruent.

*Spatio-graphic properties:* Sides of the rhombus are congruent and parallel.

*Meaning of the construction:* Theoretical: The length of the sides need not be explicitly determined and constructed. Congruent sides are the result of one pair of opposite and parallel sides intersecting with a diagonal of the rhombus.

#### **Unsuccessful construction routes**

Analysing the construction routes of unsuccessful students brings into focus the influence of the spatio-graphic properties of the object during the construction process. The students who failed to construct a spatio-graphically correct rhombus, but who had theoretically correct construction routes failed to integrate the labeling of the given angle and diagonal segment with the spatio-graphical object. Their constructions utilize the given angle MKL as an internal angle of the rhombus, and the actual lengths of the legs of the given angle as side lengths of their rhombi. We interpret this as indicative of visual reasoning at Van Hiele Level 1 – within a holistic image of a rhombus, angles are at the corners of the rhombus, and the legs of such angles are sides of the rhombus. Yet these students (5 in total) proceeded to reason successfully with the property that the diagonal of a rhombus bisects opposite angles to construct a spatio-graphically correct rhombus.



Only one student in the sample was completely unsuccessful. The student failed both logically and spatio-graphically to construct a rhombus from the given parts. The student's construction suggests that he held captive by his visual perception of a rhombus. He drew the diagonal KM in a slanted orientation to the page, then drew a segment KN parallel to the edge of the page and another segment PM visually parallel to KN and the edge of the page. He ended with an attempt to fit the given angle MKL on the diagonal, but as it did not fit the space between the diagonal and the drawn segments, he ended with a drawing (rather than a construction) of a parallelogram. Note that the student was able to copy-and-construct the given angle onto the diagonal, but he did not understand the relationship between this angle and the sides of the rhombus.

## DISCUSSION

The distinction between the use of geometric properties and spatio-graphic properties to solve the construction problem is for us a point of contention. Without access to students' verbal justification of each construction step one cannot be sure whether the step is based on spatio-graphical or theoretical information. However, our inference of the meaning of the construction based on the final construction step holds potential for interpreting the overall meaning of the construction. If the spatio-graphic success of the construction was seen as a necessary consequence of relationships between the angle and diagonal properties of a rhombus, all four side lengths did not have to be explicitly constructed. In construction routes 1 and 3, we inferred the meaning of the construction to be "find the side lengths of the rhombus", since the construction press points and arcs indicate that the constructions were completed by measuring the length of the side obtained by constructing the angle and diagonal in the correct relationship, and purposefully constructing the remaining sides. We infer that the conclusion of the rhombus in this way indicate more weight to the visually observable, holistic, spatio-graphic properties of a rhombus. Indeed, in comparison with constructions routes 2 and 4, the reasoning seems to be more on the level of reasoning with visually observable properties of the rhombus (Van Hiele Level 1 or 2) than reasoning with relationships between the properties (van Hiele Level 3).

In construction routes 2 and 4, the rhombus was completed without purposefully constructing the side lengths. The students seem to have trusted that the property of equal and/or parallel sides will follow as a result of the relationship between the diagonal and the given angle. This would be indicative of Van Hiele Level 3 reasoning. There are no press points or construction arcs to indicate measurement and construction – the relevant rays were simply extended to where they intersect the second diagonal, and the intersection points labeled. We concur with Laborde (2005) that the conclusion and verification of the constructions are based on both the theoretical and spatio-graphic properties of the constructed object and we need to interview students about their constructions to better understand the moves between theoretical and spatio-graphic information.

## CONCLUSION

We found evidence in the construction routes of the students which allows us to infer Van Hiele reasoning levels. Successful students seem to have reasoned with the relationships between the properties of the rhombus, although they showed differences in what they seem to have construed as the meaning of the construction. Constraining factors seem to emerge from both spatio-graphic and theoretical domains as well. Less successful constructions were marred according to spatio-graphical criteria, despite the theoretical correctness of the construction routes. These students also reasoned with the relationships between the properties of a rhombus, but they misappropriated spatio-graphical information like labeling on the given parts in relation to the completed rhombus. While we need further research that provides access to students verbal justifications for construction steps, we are encouraged by the geometric reasoning that is apparent in the construction routes we identified.

## REFERENCES

- Burger, W. F., & Shaugnessy, J. M. (1986). Characterising the Van Hiele levels of development in geometry. *Journal for Research in Mathematics Education*, 17(1), 31 - 48.
- De Villiers, M. (2010). Some reflections on Van Hiele theory. Paper presented at the Fourth congress of teachers of mathematics of the Croatian Mathematical Society.
- Fischbein, E. (1993). The theory of figural concepts. *Educational studies in mathematics*, 24, 139 - 164.
- Idris, N. (2009). The impact of using Geometer's Sketchpad on Malaysian students' achievement and Van Hiele geometric thinking *Journal of Mathematics Education*, 2(2), 94 - 107.
- Laborde, C. (2005). The hidden role of diagrams in pupils' construction of meaning in geometry. In J. Kilpatrick, C. Hoyles, O. Skovsmose & P. Valero (Eds.), *Meaning in mathematics education* (pp. 159 - 180). New York: Springer.
- Mariotti, M. A. (1995). Images and concepts in geometrical reasoning. In R. Sutherland & J. Mason (Eds.), *Exploiting mental imagery with computers in mathematics education* (pp. 97 - 116). Berlin: Springer.
- Martin, G. E. (1998). *Geometric constructions*. New York: Springer.
- Tapan, M.S., & Arslan, C. (2009). Preservice teachers' use of spatio-visual elements and their level of justification dealing with a geometrical construction problem. *US-China Education Review*, 6(3), 54 - 60.
- Van Hiele, P. M. (1959). Development and the learning process. *Acta Pedagogical Ultrajectna* (pp. 1 - 31). Groningen: J.B. Wolters.



# DEVELOPING RUBRICS FOR TPACK TASKS FOR PROSPECTIVE MATHEMATICS TEACHERS: A METHODOLOGICAL APPROACH

Kim Ramatlapana

University of Witwatersrand, Johannesburg

*This paper will focus its discussion on the tasks utilized in a study that seeks to understand the characteristics of prospective teachers' technological pedagogical content knowledge constructed in a GeoGebra-based environment. I will discuss a methodological approach that was adopted during the design of rubrics to code the data that will emerge from the study. Using Task 1 as an example to support the description, I will first deconstruct the task by providing a description that will elaborate the critical components and expectations of the task in constructing mathematics PTs' TPACK and then provide a description of the methodology. A detailed description of the design of the rubrics Task 1 will be discussed.*

## INTRODUCTION

Understanding teacher competence has been the focus of research for some time. The issue of teachers' knowledge of teaching for high learner achievement has contributed to the conceptualization of the term *teacher knowledge* (Beswick & Watson, 2012). Through the works of Ball, Thames and Phelps (2008), Hill, Ball and Schilling (2008) and Shulman (1986, 1987) various categories of teacher knowledge have emerged.

Technology in the teaching and learning of mathematics has been studied in several developmental research projects globally. Studies of computer use in school mathematics have largely examined innovations linked to developmental research projects. Many of these studies have investigated teacher participation and computer use in these developmental projects against the background of computer-based resources: for example, use of diverse interactive video materials to support a range of mathematical tasks at secondary level in England (Phillips & Pead, 1995); using GeoGebra to teach upper secondary level mathematics (Lu, 2008); the influence of dynamic geometry software on plane geometry problem-solving strategies (Aymemi, 2009). Jaworski (2010) has studied the challenges of using GeoGebra as a tool directed at generating conceptual understanding through exploration and inquiry for undergraduate mathematics students; Niess (2005) has investigated the development of prospective mathematics teachers' technological pedagogical content knowledge in a subject specific, technology integrated teacher preparation program.

Collaboration and partnerships on projects and studies on technology in mathematics in higher education have recently been on the rise, with developing the use of technology to support teaching and learning being identified as a priority in most of these projects.

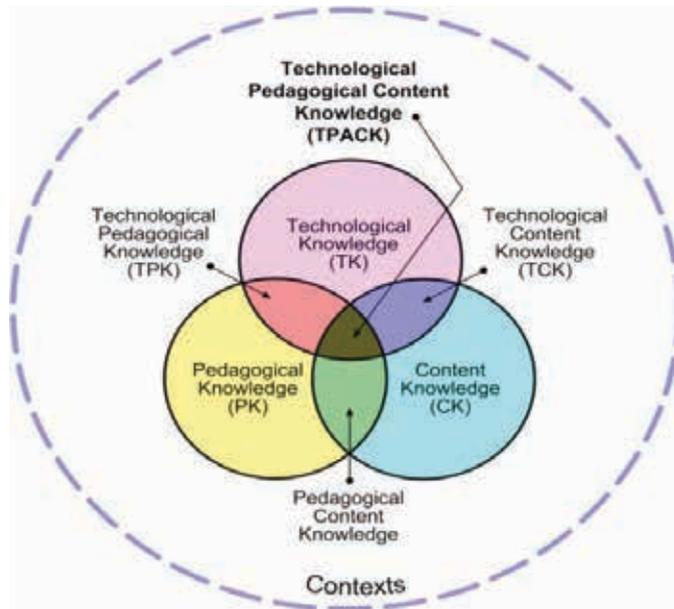
Although the technology community has advanced the benefits of integrating technology in education, there are discerning voices that have cautioned on learning in technology-based environments. For example, research has shown that technology tools can engage students in authentic learning opportunities that enhance the development of basic and higher-order skills but United Nations Educational, Scientific, and Cultural Organization (UNESCO, 2008) warns that the success to integrate lies in the ability of the teacher to effectively integrate technology into classroom lessons. Drijvers and Trouchè (2008) have acknowledged the double jeopardy of teaching and learning mathematics in a technology-based environment, given the complexities of teaching and learning and the complexities of use of the technology tool. Mathematics teachers should be knowledgeable about mathematics content, pedagogy, learners in relation to technology integration in learning. Drijvers and Trouchè (2008, p. 364) elucidate on the *double reference* phenomenon which is the double interpretation of tasks by teachers and learners giving an example where “tasks that address mathematical concepts may be perceived to address how the computer environment would deal with such a task.”

Teacher education programs are proposing that undergraduate courses in mathematics integrate technology into teaching with activities that promote mathematical thinking. In pursuit my interest in technology integration in mathematics learning I want to examine prospective teachers (*hereafter referred to as PT*) re-learning mathematics and learning to teach mathematics with technology, specifically the GeoGebra software. As mediators of mathematics learning PTs should experience technology first if they are to incorporate it into classroom mathematics learning. It is worth noting that teacher beliefs on mathematics influence their decisions on pedagogical practices. It is essential to understand the beliefs that influence teachers’ decision to use technology as these may be barriers to using technology for instruction (Hew & Brush, 2007). In the same light, intensified research is needed to improve and understand mathematics learning in technological environments; particularly, what processes and actions should be illuminated and addressed when dealing with technological artefacts in mathematics instruction. In their study on South African teachers’ use of dynamic geometry software in high school classrooms, Stols and Kriek (2011) found that teachers’ behaviour towards dynamic geometry is influenced by the perceived usefulness of technology in the classroom. Teachers’ perspectives on teaching and learning mathematics in technology-rich environments should be illuminated and explored at teacher preparation level. Niess (2005) reiterates that teacher’ decisions to implement technology into their teaching practice rests on their knowledge of technology, knowledge of mathematics, and knowledge of teaching.



## THEORETICAL FRAMEWORK

Mathematics teacher education programs need to prepare PTs so that they are able to consider the mathematics content, the technology in use and the pedagogical methods employed in teaching the content. In such programs, knowledge of technology should integrate both mathematical knowledge and knowledge about the technology tools. I contend that knowledge is derived from experience for which I conjecture that teacher knowledge is influenced and framed by teacher practical experiences with tools. Researchers in the field of technology integration employ the technological pedagogical content knowledge (*herein referred to as TPACK*) framework to study the development of teacher knowledge about technology integration (Lee & Hollebrands, 2008; Mishra & Koehler, 2006; Niess, 2005). Premised on the Lee Shulman's framework of pedagogical content knowledge (PCK), teacher knowledge for technology integration is built on the interaction among three bodies of knowledge: domain-specific content knowledge, pedagogical knowledge, and technology knowledge. I employed the technological pedagogical content knowledge (TPACK) framework as a lens to study prospective secondary mathematics teachers' knowledge development as they work on a set of geometry tasks where such tasks are designed to advance both mathematics knowledge and technology knowledge (see diagram below).



TPACK framework and its knowledge components (<http://tpack.org>)

Mishra and Koehler (2006) explicate that TPACK is the interaction of these bodies of knowledge, both theoretically and in practice, to produce the types of flexible knowledge needed to successfully integrate technology use into teaching. Mishra and Koehler (2006) explicate that TPACK is the interaction of these bodies of knowledge, both theoretically and in practice, to produce the types of flexible knowledge needed to successfully integrate technology use into teaching. The TPACK constructs as described by Mishra and Koehler (2006, p. 63) are conceptualized in the study as follows: *Content Knowledge* is knowledge of geometry concepts, theories, ideas, established practices and approaches toward developing such knowledge. *Pedagogical Knowledge* is knowledge of teaching and learning circle geometry. *Technology knowledge* is knowledge about GeoGebra and working with GeoGebra. *Pedagogical content Knowledge* is knowledge of pedagogy for teaching of circle geometry; *Technological Content Knowledge* is the understanding of how GeoGebra is best suited for addressing learning circle geometry. *Technological Pedagogical Knowledge* is the knowledge required for understanding the constraints and affordances of GeoGebra. *Technological Pedagogical Content Knowledge* is knowledge about teaching circle geometry with GeoGebra effectively.

This discussion is part of an ongoing study that intends to examine knowledge development of PTs enrolled in a secondary mathematics method course. The study focuses on mathematical thinking processes of the prospective teachers as they learn or re-learn school geometry in the GeoGebra based environment (content knowledge) and by examining what characterizes the prospective teachers' pedagogical content knowledge for teaching geometry with technology (technological pedagogical content knowledge). The knowledge development is studied in the context of investigating PTs' actions as they work on the geometry content and pedagogical tasks developed in a GeoGebra-based environment.

Often TPACK knowledge development has been studied through the use of Likert-type scales, appropriating the use of pre- and post testing to measure the development. Acknowledging the weaknesses of the Likert instrument and taking into consideration the design of the study, I decided to employ the use of rubrics. Clement, Chauvot, Philipp, & Ambrose, (2003) contend that rubrics serve a dual purpose (i) providing insights into written responses and (ii) use of numerical scores to statistically analyze responses. A rubric is a guideline that describes the characteristics of the different levels of performance used in scoring or judging a performance. An analytic rubric was preferred because it allowed for different levels of achievement of performance criteria to be determined. The PTs responses were scored according to the analytic rubric that I designed to capture TPACK-related evidence basing on the work of Miheso-O'Connor (2011) who employed the use of rubrics to measure pedagogical content knowledge proficiency in teaching mathematics. As such, the design of the rubric was guided by the question "What would the participant need to know or be able to do to successfully respond to this task?"



The rubric used specific scores based on a five-point qualitative scale (ranging from 0 to 4) to capture the PTs' proficiency in the three main knowledge domains of content, pedagogy and technology. To generate the descriptions, I conducted an item analysis of each task of the piloted tasks according to the descriptors that I developed from the three sources of evidence: TPACK constructs as conceptualized in the study, the Duval (1995) model of *perceptual* and *cognitive* perspectives on geometry reasoning and the processes of instrumental genesis in the GeoGebra construction tasks. Each task was first categorized according to the Duval's geometry apprehension and the TPACK construct that it is testing. A rubric was developed for each of the sub-task resulting with a total of 14 rubrics. An analysis of the tasks was essential in determining the reliability and validity of the items that will provide a robust evaluation of the quality of items in determining what characterizes PTs' TPACK for learning teaching geometry in technology-based environment.

### MEETING THE TASKS

The tasks selected for this study have elements of the three bodies of knowledge: content knowledge, pedagogical knowledge, and technology knowledge. Although the main emphasis of the tasks is to intertwine content, pedagogy and technology, I have decomposed the tasks based on the Stylianides & Stylianides (2010) and Biza, Nardi, & Zachariades (2007) recommended features of mathematics pedagogy and content tasks for PTs. The technology tasks are drawn from Laborde (2001) recommended features.

Stylianides & Stylianides (2010) propose that the nature of mathematics tasks for preparing teachers should engage participants in mathematics content; link mathematical ideas suggested by theory or research; and engage participants in mathematical activity from the perspective of a teacher of mathematics. Similarly, Biza, Nardi, & Zachariades (2007) suggest that the structure of tasks should explore (i) subject-matter knowledge, (ii) types of pedagogy and, (iii) types of didactical practice that describe feedback to learner's response.

The technological feature of the tasks were structured as suggested by Laborde (2001, p. 293) who categorizes *dynamic geometry environment* as:

(1) tasks for which the technology facilitates but does not change the task (e.g., measuring and producing figures); (2) tasks for which the technology facilitates exploration and analysis (e.g., identifying relationships through dragging); (3) tasks that can be done with paper-and-pencil, but in which new approaches can be taken using technology (e.g., a vector or transformational approach); and (4) tasks that cannot be posed without technology (e.g., reconstruct a given dynamic diagram by experimenting with it to identify its properties – the meaning of the task comes through dragging). For the first two types, the task is facilitated by the technology; for the second two, the task is changed by technology.

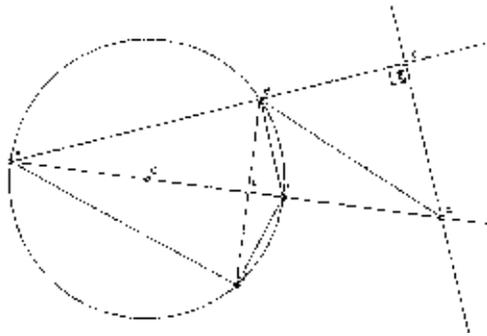
The tasks comprise of a series of content-based and pedagogical-based questions involving typical problems based on a Grade 11 geometry level, requiring the participants to construct geometrical objects with the intent to infer properties, generalities, or theorems through the different dragging modalities of GeoGebra<sup>4</sup>. In deconstructing the tasks, I have addressed three components of: (a) the critical components of the task, (b) the actions required to complete the task, and (c) the TPACK construct(s) addressed by the task or the sub-tasks.

### DECONSTRUCTING TASK 1

Task 1 comprises of content-based and technology-based questions involving typical problems based on Grade 11 geometry, requiring the PTs to interpret and construct geometrical objects with the intent to infer properties, generalities, or theorems through the different dragging modalities of GeoGebra<sup>4,2</sup> (see below). The major purpose of the task is to use multiple representations to represent and explain the task in different situations that provide opportunities for application of the cognitive apprehensions and cognitive processes for geometric reasoning.

#### TASK 1

The diagram below shows a circumscribed circle with centre S. Triangle ABC has  $AB = AC$ . Angle A is acute and AB is extended to K. AS extended cuts BC at M and the circle at H. BE bisects  $\widehat{CBK}$ . BE meets AS produced at E. AB when produced, is perpendicular to EK.



- Write down and label all the geometric shapes/figures that you see in the above diagram. E.g.  $\triangle ABC$
- Which triangles in the diagram are congruent? Justify.
- Use GeoGebra to construct the diagram.



### **The critical components of the task**

The mathematical objective of the task is to compose and decompose a shape within a given diagram by reflecting an understanding of geometrical concepts and spatial representations derived from the figure. Task 1 is based on the argument by Gagatsis, Deliyianni, Elia, Monoyiou, & Michael (2009: 37) that “geometrical figures are simultaneously concepts and spatial representations”. This argument suggests that “diagrams in two-dimensional geometry play an ambiguous role: on the one hand, they refer to theoretical geometrical properties, while on the other, they offer spatio-graphical properties that can give rise to a student’s perceptual activity” (Larborde, 2004:1). The major purpose of this task is to make a mathematical argument on generalizations and conjectures when interacting with the diagram, which Herbst (2004) purports that it provides an opportunity to make reasoned conjectures.

### **What action is required to complete the task?**

The task requires the PTs to make interpretations and constructions. The task provides an opportunity to explore the PTs’ prior knowledge regarding definitions, properties, theorems and constructions of geometric figures, deductions that can be made about these figures and the ability to transform a static drawing to a dynamic construct. Tasks 1(a) and (b) examines the ability to discriminate and recognize in the perceived figures and several subfigures and as such this task is concerned with examining the PTs’ visual spatial ability: the mental ability to manipulate objects and their parts in a two dimensional space. Task 1(b) solidifies the deductions made in (a). Task 1 (c) provides the PTs with opportunities to explore construction strategies and to solidify the idea that these constructions are based on geometric properties identified in (a) and (b). In this task PTs invent strategies for constructing a perpendicular bisector, a cyclic quadrilateral, isosceles triangle, etc, by building more sophisticated constructions, such as inscribing an isosceles triangle in a circle.

### **The TPACK construct(s) addressed by the task**

The task comprises of content-based and technology-based questions. The task is testing three TPACK constructs of CK and TCK. Tasks 1 (a) and (b) test the CK that require geometry competence. A conceptual understanding of aspects of circle geometry should be identified by making connections between concepts. Task (c) tests the TCK that requires competence to use GeoGebra to mediate geometry proficiency. The PT is required to identify the geometrical relationships between the objects created on the computer and original constructions. To successfully do the identification, PTs need to visualize the different configurations of the figures and use GeoGebra construction tools such as the ‘drag mode’ tool to explore of conjectures.

***A model solution of Task 1***

Task 1(a) requires perceptual apprehension of the figure. There are at least 17 figures that can be identified comprising a circle and composite circle, triangles and quadrilaterals, suggesting that one should be able to identify at least three types of figures:

1. Circle S, 2 semi-circles, 4 segments
2. Triangles:  $\triangle ABM$ ,  $\triangle ACM$ ,  $\triangle BMH$ ,  $\triangle MHC$ ,  $\triangle BHE$ ,  $\triangle BKE$  (all single triangles);  
 $\triangle ABC$ ,  $\triangle ABH$ ,  $\triangle AHC$ ,  $\triangle BHC$ ,  $\triangle BME$ ,  $\triangle ABE$ ,  $\triangle AKE$  (all composite triangles)
3. Quadrilaterals:  $ABHC$ ,  $BKEH$ ,  $BKEM$  (accepts kite  $ABHC$ , cyclic quad  $ABHC$ )

Task 1(b) tests knowledge of congruency.

Required to show that:

- (i)  $\triangle ABH \equiv \triangle ACH$  or equivalent

Proof:  $AB = AC$  given  
 $\hat{C} = 90^\circ$  ( $\angle$  in semi-circle)  
 $\hat{B} = 90^\circ$  ( $\angle$  in semi-circle)  
 $AH$  common

$\therefore \triangle ABH \equiv \triangle ACH$  (SAA)

(ii)  $\triangle ABM \equiv \triangle ACM$

Proof:  $AB = AC$  given

$CM = BM$  ( $\triangle ABH \equiv \triangle ACH$ )

$\hat{C}AM = \hat{C}BM$  ( $\triangle MHB \equiv \triangle MHC$ )

$AM$  common

$\therefore \triangle ABM \equiv \triangle ACM$

(iii)  $\triangle BMH \equiv \triangle MHC$

Proof:  $HB = HC$  ( $\triangle ABH \equiv \triangle ACH$ )

$CM = BM$  ( $\triangle ABH \equiv \triangle ACH$ )

$\hat{C}HM = \hat{B}HM$  ( $\triangle ABH \equiv \triangle ACH$ )

$HM$  common

$\therefore \triangle MBH \equiv \triangle MHC$



Task 1(c) requires a construction of the diagram with GeoGebra. A model solution must reflect an ability to transform the following statements from static to dynamic construction on GeoGebra:

Construction of

Triangle ABC

AS extended cuts BC at M and the circle at H.

BE bisects  $\widehat{CBK}$ .

BE meets AS produced at E.

AB when produced, is perpendicular to EK

## THE RUBRICS

Following is a discussion of the intensity of the methodological process employed to develop the rubrics. The rubrics had to be specific and explicitly address the expectations of the tasks. However, I acknowledged that the rubrics at this stage should be flexible considering that I was developing description of anticipated typical responses that might be availed. As such, the constructed rubrics are to be a guideline to analyzing the PTs responses. The descriptions developed were built from the expected ideal solutions devised in the memo. I utilized a five-point qualitative scale ranging from a score of 0 for non response and/or incorrect response to a score of 4 for a correct response. I used a reverse method in determining the description starting with level 4 building down to level 0. The description for level 4 was based on the ideal correct solution, where all traits in the description are realized. In some instances, examples had to be given as a guide for some descriptions to make clear where certain responses will fit.

### Task 1(a)

This task tested PTs' geometry content knowledge. The PTs were required to "Write down and label all the geometric shapes/figures that you see in the above diagram". To avoid misunderstandings an example was indicated to lead the respondent on the expected answer. In levels 4 – 1, the descriptions reflect that respondent will correctly identify and label the figures with at least mentioning the three figures (see rubric 1(a)). I expect the PTs' to know basic figures i.e. circle, triangle, quadrilaterals. Despite this, Level 1 caters for responses that I anticipate mention 2 figures correctly regardless of the type of figure. I considered that labeling could be a constraint to some respondents. There are at least 17 figures that one can recognize in the perceived figures and several subfigures, an interval of number of figures had to be determined for the 4 levels. As mentioned it was justified that the lowest number of figures should be 3 and the maximum for a response that considered the figures built from the three basic figures is 17. However, an exceptional case would be an inclusion of semi-circles and circle segments. This statement qualifies the *at least 17* figures identified.

level	description
0	No shape/figure identified
1	Correct identification and labeling of at least 2 figures even if similar e.g. all triangles
2	Correct identification and labeling of 3 - 9 figures including three major shapes: circle, triangle, quadrilaterals
3	Correct identification and labeling of 10 - 16 figures including three major shapes: circle, triangle, quadrilaterals
4	Correct identification and labeling of at least 17 figures including three major shapes: circle, triangle, quadrilaterals

Table 1: Rubric 1(a)

### Task 1(b)

This task tested PTs' geometry content knowledge. The PTs were required to show and justify "which triangles are congruent". In levels 4 – 1, the descriptions reflect that the respondent will correctly identify the congruent triangles basing on the recognition that AH is the diameter of the circle (see rubric 1(b)). The mathematical statement given in the responses for these levels should reflect both the visualization and reasoning process enacted. However, we noted that a correct identification or configuration of the diagram to show congruency may not necessarily be aligned with the correct reasoning or justification. As such, the explanations were coded with respect to the levels as correct, incomplete correct, faulty, and no explanations. For instance, Level 3 differs with level 4 in that the level 3 response provides a correct response with incomplete explanations.

level	Description
0	incorrect identification of congruent $\Delta$ s or no response
1	Correct identification of at most 3 congruent triangles; no explanations
2	Correct identification of at most 3 congruent triangles ; Faulty explanations
3	Correct identification of at most 3 congruent triangles ; incomplete correct explanations
4	Correct identification of at most 3 congruent triangles. Correct explanations using geometric reasoning, recognizing in reasoning that AH is diameter.

Table 2: Rubric 1(b)



**Task 1(c)**

This task tested PTs’ geometry technological content knowledge. The PTs were required to “Use GeoGebra to construct the diagram”. In this task there is interplay between GeoGebra and geometry knowledge. The intention is for the descriptions to capture both knowledge of GeoGebra and geometry knowledge. The response for the task requires a proper use of GeoGebra, suggesting that in constructing the diagram with GeoGebra, there are three possibilities; a correct construction, an incorrect construction or no construction noting the level of geometry knowledge (see rubric 1(c)). A level 4 description indicates a response that shows a correct construction at a glance, suggesting that during the construction process, a complete exploitation of the affordances of GeoGebra was realized, resulting with short concise sequence of construction. A Level 3 description shows a response that correctly constructs the diagram but uses a long concise sequence of construction. A level 2 description is for an incorrect disjointed construction that indicates less exploitation of affordances of GeoGebra. At this level there is no systematic approach to the construction with a possibility of disorientation when a point is dragged. A systematic approach would optimally use GeoGebra as a dynamic geometric tool. At level 1 the response indicates an attempt to construct but not necessarily the required diagram, reflecting some technical knowledge but lack of geometry knowledge.

level	description
0	In ability to use GeoGebra
1	Some figure drawn, missing other details e.g. $\triangle ABC$ not isosceles
2	incorrect disjointed construction, less exploitation of affordances of GeoGebra, no systematic approach to construction, possibility of disorientation when point is dragged
3	Correct construction at a glance, complete exploitation of affordances of GeoGebra, long sequence of construction
4	Correct construction at a glance, complete exploitation of affordances of GeoGebra, short concise sequence of construction

Table 3: Rubric 1(c)

## CONCLUSION

This paper provided a methodology for developing rubric to analyze PTs' TPACK. The development of rubric is a lengthy process that requires a negotiation that would cater for all possible strategies for the solutions. Distinguishing between cases required a negotiation between the theoretical to the practical. This process necessitated mediation between item analysis of the tasks and descriptions of the rubrics that focus on the TPACK constructs. The tasks and the rubrics were rigorously tested for coherence, reliability, and validity during this process. To test for validity and reliability I ensured that the description were explicit and appropriate for each level. There was also a need for coherence between the expectation of the task and the rubric descriptions. I acknowledge that at this level of constructing descriptions without the data at hand, rubrics constructed should be flexible to accommodate all possible responses.

## REFERENCES

- Aymemi, J. (2009). Influence of dynamic geometry software on plane geometry problem-solving strategies. Unpublished PhD thesis. Universitat Autònoma de Barcelona: Spain.
- Ball, D. L., Thames, M.H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? . *Journal of Teacher Education*, 59(5), 389-407.
- Beswick, K., Callingham, R., & Watson, J. (2012). *The nature and development of middle school mathematics teachers' knowledge*. *Journal of Mathematics Teacher Education*, 15, 131-157.
- Biza, I., Nardi, E., & Zachariades, T. (2007). Using tasks to explore teacher knowledge in situation-specific contexts. *Journal of Mathematics Teacher Education*, 10, 301-309.
- Clement, L., Chauvot, J., Philipp, R., & Ambrose, R. (2003). A method for developing rubrics for research purposes. In N. A. Pateman, B. J. Dougherty, & J. T. Zilliox (Eds.), *Proceedings of the 2003 joint meeting of PME and PMENA* (Vol. 2, pp. 221-227). Honolulu: CRDG, College of Education, University of Hawaii.
- Duval, R. (1995a). Geometrical Pictures: Kinds of Representation and specific Processings. In R. Sutherland & J. Mason (Ed.). *Exploiting Mental Imagery with Computers in Mathematic Education*. Berlin: Springer p. 142-157.
- Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics* (2006) 61: 103-131
- Gagatsis, A. A., Deliyianni, E. E., Elia, I., Monoyiou, A., & Michael, S. (2009). A structural model of primary school students' operative apprehension of geometrical figures. In M. Tzekaki, M. Kaldrimidou & C. Sakonidis (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. XXX-YYY. Thessaloniki, Greece: PME
- Herbst, P. (2004). Interactions with diagrams and the making of reasoned conjectures in geometry. *ZDM*. 36(5), 129-139
- Hew, K. F., & Brush, T. (2007). Integrating technology into K-12 teaching and learning: current knowledge gaps and recommendations for future research. *Education Tech Research Dev*, 55, 223-252.
- Hill, H., Ball, D. L., & Schilling, S. (2008). Unpacking "pedagogical content knowledge": Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education*, 39(4), 372-400.
- Jaworski, B. (2010). Challenge and support in undergraduate mathematics for engineers in a geogebra medium. *MSOR Connections*, 10(1), 10-14.



- Laborde, C. (2001). Integration of technology in the design of geometry tasks with CabriGeometry. *International Journal of Computers for Mathematical Learning*, 6(3), 283-317.
- Lee, H., & Hollebrands, K. (2008). Preparing to teach mathematics with technology: An integrated approach to developing technological pedagogical content knowledge. *Contemporary Issues in Technology and Teacher Education*, 8(4). Retrieved from <http://www.citejournal.org/vol8/iss4/mathematics/article1.cfm>
- Lu, Y. A. (2008). Linking Geometry and Algebra: A multiple-case study of Upper-Secondary mathematics teachers' conceptions and practices of GeoGebra in England and Taiwan. Unpublished MPhil thesis. University of Cambridge.
- Miheso-O'Connor, M. (2011). Proficiency in pedagogical content knowledge for teaching mathematics: Secondary School mathematics teacher's interpretation of students' problem-solving strategies in Kenya. Leipzig: VDM Verlag Dr. Müller
- Mishra, P., & Koehler, M. J. (2006). Technological Pedagogical Content Knowledge: A new framework for teacher knowledge. *Teachers College Record*, 108(6), 1017-1054.
- Niess, M. L. (2005). Preparing teachers to teach science and mathematics with technology: Developing a technology pedagogical content knowledge. *Teaching and Teacher Education*, 21, 509-523
- Phillips, R., Gillespie, J., & Pead, D. (1995). Evolving strategies for using interactive video resources in mathematics classrooms. *Educational Studies in Mathematics*, 28(2), 133-154.
- Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14.
- Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, 57(1), 1-22.
- Stols, G., & Kriek, J. (2011). Why don't all maths teachers use dynamic geometry software in their classrooms? *Australasian Journal of Educational Technology*, 27(1), 137-151.
- Stylianides, G. J., & Stylianides, A. J. (2010). *The Mathematical Preparation Of Teachers: A Focus On Tasks*. Paper presented at the CERME 6: Working Group 10. Retrieved from [www.inrp.fr/editions/cerme6](http://www.inrp.fr/editions/cerme6)
- UNESCO. (2008). ICT competency standards for teachers. Retrieved from [www.unesco.org/en/competency-standards-teachers](http://www.unesco.org/en/competency-standards-teachers)

## DATA IS KEY

Dave Rowley

BLOODHOUND SSC Education Director - South Africa, Kimberley

### *Extended abstract*

The Bloodhound education programme is available to any school or teacher in South Africa and we will be sharing all the research, design, build and testing of the car, including where the engineering team got it wrong the first time!

This motivation to share all the project information is unique in the field of advanced engineering and will ensure schools around the world have access to all the car run data. Thanks to MTN and Poynting Antennas this data will be available in virtual real time as they have developed new technology to capture and communicate all the sensor data and HD camera material via the World Wide Web. Graphs in school will never be the same again!

‘Data is key to pushing the boundaries – work out what you need and build it in from the start’. That was the advice we received from former NASA astronaut Neil Armstrong when he visited the BLOODHOUND Technical Centre back in 2010. This was very much our plan, but it was great to hear it confirmed by the first man to walk on the Moon. The reality seemed a long way off at the time, but suddenly it’s here and we’re building a myriad of sensors into BLOODHOUND SSC.

The plan for BLOODHOUND SSC involves around 400 high-speed sensors, measuring everything from air pressure in over 100 places to validate the airflow modelling, through to structural loads, and even Andy Green’s heart rate while he’s driving at 1 610 km/h . Each one of these sensors will be recorded at 500 Hz (500 times a second), so that the engineering team can analyse the data in incredible detail.

At peak speed in BLOODHOUND SSC, the air will be tearing past Andy’s office window at 450 metres/second, so working out what happens on each run will be vital. It was this sort of approach that got Neil Armstrong to the moon in 1969.

The previous World Land Speed Record, back in 1997 when Thrust SSC went supersonic, was the first time that this sort of data-intensive approach had been used for record breaking. The technology was a bit more basic, with just over 100 sensors measuring temperatures, pressures and loads at only 80 Hertz, or less. That was however enough for the Thrust team to be able to keep the car safely on the ground, and to set the first supersonic World Land Speed Record.



Data helps with more than just keeping the car on the ground. The straight line performance of the car – in other words, how fast it will actually go – can also be determined from the data. Since this is the whole aim of the World Land Speed Record, it's important stuff. But it's more even than that. The BLOODHOUND team's performance expert, Ron Ayers, has spent over 2 decades analysing performance data from every available source, going back to the 1920s. As a result, he can predict more accurately than anyone alive just how fast a Land Speed Car will go, and how long it will take to stop.

Technology has moved on considerably since the team's last successful attempt at the World Land Speed Record and this allows us to share the run data with education across the globe. What is crucial however is that teachers in South Africa are aware of this exciting adventure that is taking place on their doorstep and help us to identify the best way of using all the run data in the classroom. The development of a car capable of 1 690 km/h (1 050 mph) is down to the use of mathematics and the real world data obtained from BLOODHOUND SSC will hopefully aid the demystification of the subject in the classroom. As a result of this presentation we hope to have many more maths teachers registered to the BLOODHOUND education programme.

## REFERENCES

<http://www.bloodhoundssc.com/>

<http://www.bloodhoundssc.com/news/ten-astounding-facts-about-bloodhound-ssc>

<http://www.bloodhoundssc.com/project/facts-and-figures>

## GEOMETRICAL CONCEPTS IN REAL-LIFE CONTEXT: A CASE OF SOUTH AFRICAN TRAFFIC ROAD SIGNS

Sibawu Witness Siyepu and Thembinkosi Mtonjeni

Cape Peninsula University of Technology

*This article explores possible geometrical concepts that may be taught through the use of South African traffic road signs. The study employed a qualitative interpretive case study approach. Data was collected by means of picking up certain South African traffic road signs from internet websites. Due to limitation of the study the focus was only on control, information and hazard warning signs. Realistic Mathematics Education approach was used as a theoretical framework underpinning the study to analyse the data collected. Findings of this study reveal that geometrical concepts such as triangles, circles, rectangles, lines, squares, areas and perimeters can be taught using South African traffic road signs. All the concepts mentioned above can be taught with their properties. This study recommends the use of South African traffic road signs to demonstrate the importance of geometry in a real-life context.*

**Key words: Realistic Mathematics Education Approach; Real-life context; Geometrical concepts; South African traffic road signs**

### Definitions of key words

1. Realistic Mathematics Education (RME) approach is a mathematical approach using a theory which emphasises that the teaching and learning of mathematics must be connected to reality as well as made a human activity (Freudenthal, 1991).
2. Real life context refers to a use of knowledge connecting content to a desired real-world outcome to demonstrate its practical value.
3. Geometrical concepts are concepts that are developed to lay a foundation for the understanding of practical or Euclidean geometry such as lines, angles and shapes.
4. South African traffic road signs are signs designed to regulate traffic in such a way that traffic flow and road traffic safety are promoted.

### INTRODUCTION AND BACKGROUND

In the recent past Euclidean Geometry proved to be a huge challenge to many school mathematics students in South Africa. Its exclusion from the secondary school curriculum presented a problem to students registering for engineering courses at university. Nonetheless, the space of poor mathematics results at school prompted the politicians to influence curriculum designers and policy makers to make Euclidean Geometry optional in the curriculum. The purpose of optionalising Geometry was to pretend as if the mathematics results have improved.



Specialists of various disciplines such as engineering, built and construction, and architecture together with mathematicians criticised the issue of optionalising geometry as a way of marginalising South African students from the development of advanced understanding of mathematics (Jansen & Dardagan, 2014). The on-going poor quality in mathematics teaching and learning in South Africa is the ‘most important obstacle to African advancement’ (Centre for Development and Enterprise, 2004, p. 239). At the heart of this concern is the fact that the present education system has disadvantaged learners by failing to meet their educational needs, especially regarding mathematics (Evoh, 2009).

Padayachee et al. (2011) state that “it has been our observation in lecturing first-year mathematics students at a large metropolitan university in the Eastern Cape in South Africa that many first-year students are under-prepared for [university] mathematics” (p.1). This under-preparation may be attributed to the shortage of adequately qualified teachers and lack of resources at schools attended by the majority of learners (Adler, Brombacher & Human, 2000).

Despite introducing geometry back to the school curriculum it remains a threat to many students, teachers, curriculum advisors, and to a number of educational officials in South Africa (Siyepu, 2012). This is supported by Jansen and Dardagan (2014) who maintain that “school experts have warned this year’s matriculation results could drop as many Grade 12 teachers in state schools of South Africa struggle to prepare learners to write a compulsory section on Euclidean Geometry for the first time” (p.1). Euclidean Geometry, formerly included in the mathematics curriculum before being thrown out by the Basic Education Department in 2008, involves the properties of shapes (Jansen & Dardagan, 2014). The content of Euclidean Geometry was reintroduced in 2010 when universities warned that matriculants signing up for engineering and related courses were not coping because they had no knowledge of Euclidean concepts (*ibid*).

Several researchers attempted to develop different strategies and approaches that may improve students’ understanding in geometry. Researchers such de Villiers (1997) criticised the traditional approach dominated by teacher tells as the main cause of the poor understanding of geometry among South African students. Inductive approach and/or investigative approach were proposed by other mathematics educators such as Serra (1997) to replace the traditional teaching approach, which emphasises the mastery of content without the development of skills and students’ critical thinking capabilities.

One of the specific skills espoused by the National Curriculum and Assessment Policy Statement of South Africa is to build awareness of the important role that mathematics plays in real-life situations including personal development of the learner as well as its role in career orientation (South African Department of Basic Education, 2010). The South African Department of Basic Education introduced Mathematical literacy which focuses heavily on the use of mathematics in real-life contexts (Julie, Holtman & Mbekwa, 2011).

Julie et al. (2011) noted that teachers struggle to realise this feature of the curriculum. As a result there is no much emphasis on the use of mathematics in real-life situation in a South African context (Julie et al., 2011). One other important factor often ignored is to improve the teaching and learning of geometry in a South African context. This can be achieved by relating the school curriculum with real-life context. Active engagement of learners and development of their critical problem solving skills using real life situation will demystify the myth that mathematics exists only inside the classroom.

The essence of this study is to explore geometrical concepts that are applied in the development of South African Traffic Road using control, information and hazard warning signs.

### **THEORETICAL FRAMEWORK**

The learning of mathematics in South Africa is characterised by memorisation of rules, signs and procedures. This imposes limitation in terms of learners' development of conceptual and problem solving skills. Several researchers have queried the lack of focus on the students' developmental understanding of concepts. This study focuses on the development of geometrical concepts using South African Traffic Road signs in a mathematics classroom. The authors employed a Realistic Mathematics Education approach as a theoretical framework underpinning this study.

Wubbels, Korthagen and Broekman (1997) assert that in a Realistic Mathematics Education approach:

The inquiry process is characterised by the consecutive steps of translating a real world problem into a mathematical problem, the analysis and structuring of such a problem, the creation of a mathematical solution, the translation of this solution to the real world and the reflection on the merits and restrictions of the solution, which can be followed by a next cycle in which the translation of the problem is refined, generalised or otherwise changed. (p.7)



The philosophy underpinning Realistic Mathematics Education (RME) is that students should develop their mathematical understanding by working from contexts that make sense to them (Dickinson & Hough, 2012). This theory originated from the Freudenthal Institute (FI) at University of Utrecht in the Netherlands. This institute was set up in 1971 in response to a perceived need to improve the quality of mathematics teaching in Dutch schools. This initiative led to the development of a research strategy and to a theory of mathematics pedagogy called “Realistic Mathematics Education” (RME) which is now used almost globally (Dickinson & Hough, 2012).

### **Evidence for the effectiveness of RME**

In international mathematics tests, the Netherlands is now considered to be one of the highest achieving countries in the world (Dickinson & Hough, 2012). They assert that “teachers using RME report that it enables more students to understand mathematics and to engage with it” (p. 6). They also noted that, the performance of the Netherlands in international comparisons of mathematical attainment has been consistently strong over the recent years. Results from the two major international comparative studies (Programme for International Student Assessment (PISA) and the Trends in International Mathematics and Science Study (TIMSS)) indicated that those students who had experienced RME were not only more likely to solve a problem correctly, but showed considerably more understanding through their ability to explain their strategies (Dickinson & Hough, 2012).

### **RESEARCH METHODS**

This study is located within the interpretative qualitative research paradigm. Qualitative research is an exploratory approach, which emphasises the use of open-ended questions and probes, which give participants an opportunity to respond in their own words (Devetak, Glazar & Vogrinc, 2010). This research is a single case study, which focuses on the use of South African Traffic Road Signs to make sense of geometrical concepts in a mathematics classroom. The study is limited to three kinds of South African Traffic Road signs, namely: control, information and hazard warning signs.

## Data collection

The researchers collected data by means of document analysis. This entails the analysis of South African Road traffic signs in particular control, information and hazard warning signs which were searched from the Internet, and were printed out to be used in data analysis as shown in figures 1, 2 and 3 below:

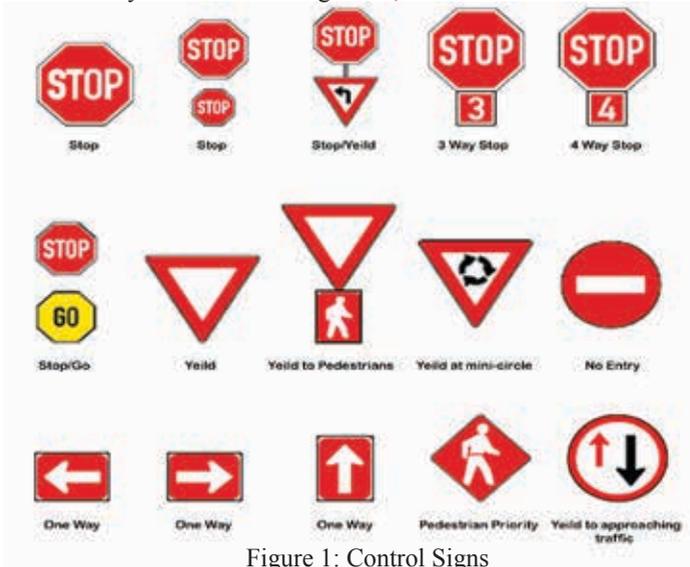


Figure 1: Control Signs

**STOP SIGNS** are regulatory devices often mounted on the roadside near the intersection to inform the driver to stop temporarily before proceeding, provided there is no danger of a crossing object ahead. These signs are indicated by octagonal shapes that means regular shapes with eight sides equal. **STOP/YIELD** is indicated by an octagonal shape and a triangular shape with a red edge and interior white triangular shape. The three and four way stops are made up of an octagonal shape with a squared based shape. **STOP** and **GO** is made up of octagonal shapes. They only differ in colours **STOP** is red and **GO** is yellow. **YIELD SIGNS** are made up of triangles. **NO ENTRY** is made up of a red circle and a white rectangle inside the circle. **ONE WAYS** are made of rectangles shaded in red colour and with white arrows inside. Arrows resemble rays. **PEDESTRIAN PRIORITY** is made of a red square with a human picture. Yield as you approach a traffic circle is made of a circle and two arrows one black downward and a red one upward.



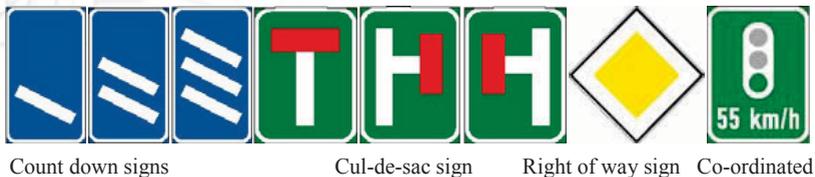


Figure 2: Information Signs

Information signs inform the people about certain objects and their use. This includes giving information in advance about road condition. Figure 2 shows four types of information signs such as **COUNT DOWNSIGNS**, **CUL-DE-SAC**, **RIGHT OF WAY** and **COORDINATED SIGNS**. Count down signs are made up of rectangles shaded blue with one up to three rectangular blocks inside shaded white. Cul-de-sac signs are made up of rectangles shaded green with T-shaped red rectangles. Right of way is made of yellow diamond like square. Co-ordinated traffic signs are made up of three circular shapes in a rectangular shape.



Figure 3: Hazard Warning Signs

**HAZARD WARNING SIGNS** indicate danger or obstacle lying on the road. They are: **DANGER PLATE**, **SHARP CHEVRON** and **ARRESTOR BED**. Danger plate signs show tessellated triangles and regular parallelograms. Sharp chevron signs show tessellated rectangles. Arrestor bed is made up of tessellated red and white squares in a rectangular block.

### Data analysis

Document Analysis, particularly interpretative analysis, aims to capture hidden meaning and clarify ambiguity. It looks at how messages are encoded, latent or hidden. Researchers' interpretation should be informed by the awareness of who the members of the audience are. The target audience for this study is teachers, lecturers, subject advisors and other parties interested in the teaching of mathematics using real-life contexts.

The two authors gathered together to find out about geometrical concepts that can be taught using the identified South African Traffic Road signs as in figures 1, 2 and 3 above. Their analysis was guided by the following questions: “What are geometrical concepts that are resembled by the South African Traffic Road signs reflected in warning, information and hazard warning signs as shown in figures 1, 2 and 3 above?” and “How can teachers link these signs with geometry lessons in their mathematics classrooms?”. The researchers analysed the collected data step by step from figure 1 to figure 3 writing each concept as it emerges from their discussion. Lastly, they decided on what should be written in a research report.

## DISCUSSION

The South African Traffic Road signs are made up of various geometrical shapes. These signs can be used in the classroom, firstly for the recognition or visualisation as the first level of van Hiele’s theory in the learning of geometry. Secondly these signs can be used to develop learners understanding of the features of geometrical shapes mostly two-dimensional shapes.

### Discussion with respect to Figure 1

Figure 1 comprises of triangles, rectangles, circles, octagonal shapes and rectangular arrows. The **NO-ENTRY** sign can be used to determine:

1. The circumference as the perimeter of a circle;
2. Area of a circle as  $A = \pi r^2$ ;
3. The area of an inside rectangle;
4. The area of the portion marked red which is equal to the area of the whole circle minus the area of the white rectangle inside.

In other shapes such as octagonal the properties of regular octagon are:

1. All sides are equal;
2. Each angle measures  $135^\circ$ ;
3. It introduces triangulation of shapes.
4. This is done to investigate the method of developing formula for the area of a shape.

Through rectangles and triangles learners are exposed to the properties of various shapes. They develop a skill in counting the number of sides, an area of shapes, a rectangle and a triangle. Learners can also be introduced to the understanding of special triangles such as ‘isosceles’, ‘scalene’ and ‘equilateral triangles’. In a rectangle, learners can be introduced to the use of protractors to measure angles and to special properties such as one angle measures  $90^\circ$ . They learn that diagonals are equal; diagonals bisect each other; and opposite sides are equal.



With yields at mini-circle, learners can be exposed to the calculation of the area of the outer triangle shaded red by subtracting the area of the triangle shaded white. These aspects reinforce the understanding of the areas of various shapes including triangles, rectangles and circles. Yields signs are usually found when approaching traffic circle. Concentric circles can be introduced as two or more circles with a common centre. These signs may be used to reinforce the area of a circle, and for a calculation of the area shaded red. All the components and properties of a circle can be developed, namely: diameter, radius, centre, segment, sector, tangent, secant, and a chord. Effectively, learners are given an opportunity to explore ways of developing various conjectures and theorems from these shapes.

### **Discussion with respect to Figure 2**

Figure 2 deals with geometrical concepts contained in the information signs. From count down signs rectangles that are identified assist in the teaching of parallel lines, their properties and the definition of parallel lines. Interpretation of rectangles brings about the understanding of congruent shapes. Then, in cul-de-sac signs co-interior angles on the same side of a transversal, right angles, perpendicular lines and adjacent angles are identified. Moreover, supplementary adjacent angles can be identified. In a right of way sign squares and their properties can be identified.

### **Discussion with respect to Figure 3**

Danger plate signs show tessellated triangles and regular parallelograms (Figure 3). This refers to a pattern made up of identical shapes which are fitted together without any gaps, and these shapes should not overlap. Sharp chevron signs show tessellated rectangles. Arrestor bed is made up of tessellated red and white squares in a rectangular block. Tessellation is used to calculate the number of tiles to be fitted in various shapes on the floors and walls (reference to another real life situation).

## **CONCLUSION**

This study revealed that several geometrical concepts can be identified from South African Traffic Road signs. These concepts can be taught through designing mathematical lessons based on South African Traffic Road signs. The key shapes in the study of geometry emerge in the designing of South African Road traffic signs. Such shapes are: circles, rectangles, triangles and octagon. The authors explored such signs in the South African Road traffic context. From these core shapes (as discussed above) properties may be included when developing geometry lessons. For instance, in a circle, components such as diameters, radius, centres, tangents, chord, secant, segment, and sectors can be taken further with the aim of advancing the students ability to develop conjectures and also by formulating strategies of developing various Euclidean geometry theorems.

In a rectangle, the properties such as diagonals, right angled triangles, hypotenuse, application of Pythagoras theorem, and general properties of a rectangle can be used to develop theorems based on rectangles and parallelograms. Furthermore, in triangles many properties can be developed such as similar triangles and their properties; different types of special triangles such as scalene; equilateral; and isosceles triangles. This may also lead to a discussion of various types of lines such as ray, line segment, parallel lines, transversals, intersecting lines and perpendicular lines.

The main focus of the study was on the exploration of geometrical concepts that emerged from the analysis of South African road traffic signs. Although the study has some limitations in the development of geometry concepts but the most interesting aspect is to give students awareness about the application of geometry concepts in the designing of Traffic signs in a South African context. This encourages teachers and stake-holders to relate their teaching with every day-life situation – enhancing learners’ conceptual and critical thinking skills.

## References

- Adler, J., Brombacher, A., & Human, P. (2000). *Submission by the mathematics education community to the Council of Education Ministers*. Retrieved July 20, 2011, from <http://www.sun.ac.za/mathed/AMESA/NGO.htm>
- Centre for Development and Enterprise. (2004). *From laggard to world class: Reforming maths and science education in South Africa's schools*. Johannesburg: Centre for Development and Enterprise. Available at [http://www.cde.org.za/attachment\\_view.php?aa\\_id=208](http://www.cde.org.za/attachment_view.php?aa_id=208)
- Devetak, I., Glazar, S.A., & Vogrinc, J. (2010). The role of qualitative research in science education. *Eurasia Journal of Mathematics, Science and Technology Education*, 6(1), 77-84.
- De Villiers, M.D. (1997). The future of secondary school geometry. *Pythagoras*, 44 (12), 37-54.
- Dickinson, P. & Hough, S. (2012). Using Realistic Mathematics Education in UK classrooms. In P. Nicholson. ISBN: 978-0-948186-24-0
- Evoh, J.C. (2009). The role of social entrepreneurs in deploying ICTs for youth and community development in South Africa. *Journal of Community Informatics*, 5(1). Available from <http://ci-journal.net/index.php/ciej/article/view/459/438>
- Jansen, L. & Dardagan, C. (2014). Change in maths may hit matric results. Independent Oline News. 03 March 2014
- Julie, C. Holtman, L & Mbekwa, M (2011). Rasch modelling of mathematics and science teachers’ preferences of middle situations to be used in mathematical literacy. *Pythagoras*, 32(1), Art # 13, 8 pages. <http://dx.doi.org/10.4102/pythagoras.v32i.12>
- Padayachee P, Boshoff M, Olivier W & Harding A 2011. A blended learning Grade 12 intervention using DVD technology to enhance the teaching and learning of mathematics. *Pythagoras*, 32(1), Art #24, 8 pages. doi:10.4102/pythagoras.v32.i1.24
- Serra, M. (1997). *Discovering geometry: An inductive approach* (2nd ed). California: Key Curriculum Press.
- Siyepu, S.W. (2012). Some mathematical possibilities in the building of a rondavel. In S. Nieuwoudt; D. Laubscher & H. Dreyer; Mathematics as an Educational task. Proceedings of the 18<sup>th</sup> Annual National congress of The Association for Mathematics Education of South Africa, pp 322-340, North West University, Potchefstroom.



South African Department of Basic Education (2010). National Curriculum Statement; Department of Basic Education; Pretoria; Government Print

Wubbels, T., Korthagen, F., & Broekman, H. (1997). Preparing teachers for realistic mathematics education. *Educational Studies in Mathematics* **32**: 1–28,

## MATHEMATICAL QUESTION TYPES

Charles R. Smith & Cyril Julie

Western Cape Education Department & University of the Western Cape

*Questions are of cardinal importance in mathematics and its teaching mathematics. Various classification schemes of mathematical question types exist and are in use. Most of these have little practical currency for the day-to-day practice of teaching. We discuss the development of a scheme of question types that evolved from the way teachers talk about examination and other assessments. The way teachers are beginning to use the scheme is demonstrated and it is concluded that such schemes evolving from the issues and dilemmas teachers face have high possibility to contribute towards more productive teaching and meaningful learning.*

### INTRODUCTION

School-based assessment is an important component of the teaching and learning accountabilities within the schooling system. Setting quality assessments and tests is not an easy task, as is evident from the Department of Basic Education's (DBE) report on the moderation of school-based (DBE, 2013:43) assessments. Consequently the DBE suggests the development of quality assessment tasks which will serve as exemplars to guide teachers for setting school-based assessments.

As a response this article provides a conceptual scheme of question types to assist teachers and other developers of assessment tasks.

From another angle, questioning plays an important role in the advancement of mathematics. Brown and Walter (1983: pp. 2-3) draws attention the primacy of questions when they relay how after hundreds of years of attempting to prove Euclid's fifth postulate, mathematicians got a handle on it by considering the question "*How can you prove the parallel postulate from the other postulates or axioms?*" (Italics in original) instead if just focusing on the proving of the fifth axiom.

Many typologies for classifying question types for school mathematics exist. Widely known and used are Bloom's revised and the SOLO taxonomies. In many instances these taxonomies are adapted to suit particular contexts and needs. For example, the scheme used for classifying question types to be included in school Mathematics examinations as per the Curriculum and Assessment Policy Statement (CAPS) is an adaptation of Bloom's revised taxonomy. The expectation is that teachers will use the level-based scheme of question types for the setting of school-based assessments.



Another characterisation of questions is classifying them as open or closed questions. Regarding open and closed questions Boaler & Brodie (2004) concluded that regardless of the teaching approach adopted by teachers these question types were present in the approach adopted by teachers. Watson and De Geest (2012: 227) proffers that the preponderance of closed questions does not inhibit improvement of learning.

Although the typologies are useful and productive for the design of examinations and other assessments, our experience with teachers is that they particularly find the typologies not user-friendly for their practice. A plausible reason is that these typologies are normally distributed through a “research discourse generated within education faculties [and other research environments]...not readily [accessible] by the majority of South African teachers”. (Wright, 2013: 23). He calls for “a situationally committed mode of research discourse...Discourse 4, addressing the needs of teachers.” (pp. 26-27).

We concur with this sentiment and in our quest to address needs of teachers, we developed with mathematics teachers, a scheme of question types which we contend are more easily understood by them and carries more practical currency for them.

### EVOLUTION OF THE SCHEME

Our work with teachers is described in various publications (e.g. Julie, 2012; Julie 2013) Central to our work is the transformation of ideas offered by teachers, in their practice discourse, into classroom implementable modalities.

This particular scheme flowed from discussions during an in-service session on examination-setting and marking, the 2013 Annual National Assessments (ANA) and the 2013 National Senior Certificate examination for Mathematics. In particular the comment “When questions are turned around, learners find them difficult to deal with”, focused our attention. “Questions turned around” has a particular meaning in teacher discourse. In the 2013 ANA Grade 9 Mathematics test, for example, question 1.4 is an example of a “turned around” question type. Its formulation is follows:

1.4 Given the expression  $\frac{x-y}{3} + 4 - x^2$

Circle the letter of the incorrect statement

- A The expression consists of 3 terms
- B The coefficient of  $x$  is 1
- C The coefficient of  $x^2$  is -1
- D The expression contains 2 variables

The “turned around” nature is embedded in the dominant school mathematics culture where for questions in elementary introductory algebra, the normal formulation is that learners have to write down the coefficients of the terms. A similar notion of a “turned around problem” was expressed about the statistics item in the 2013 NSC Mathematics Paper 2 as presented in the figure below.

Mathematics/P2

5  
NSC

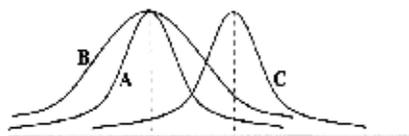
DBE/November 2013

**QUESTION 4**

The Grade 10 classes of three schools wrote a term test. All three schools have the same number of learners in Grade 10. The results of the tests have been summarised in the table below.

	SCHOOL A	SCHOOL B	SCHOOL C
Mean	9,8	9,8	14,8
Standard deviation	2,3	3,1	2,1

The distribution of the results is shown in the diagram below.



- 4.1 In which school (A, B or C) is the majority of the results more widely spread around the mean? Give a reason for your answer. (2)
  - 4.2 What is the difference in the spread around the respective means of the marks in School A and School C? (1)
  - 4.3 Explain how the marks of School A must be adjusted to match the marks of School C. (2)
  - 4.4 If each mark in School C is lowered by 10%, explain the effect it will have on the mean and standard deviation of this school. (2)
- [7]

Normally questions dealing with this topic would require examinees to calculate the means, standards deviations without really interpreting these values. Another comment frequently made teachers is that “Children do not understand concepts”. This “do not understand concepts” covers a wide range of issues. Manifestations in learner work such as slips, misconceptions, manipulative errors, incorrect application of mathematical conventions and flexible ways of understanding mathematical concepts are some of the issues.



In terms of the focus of this paper, the quest was how questions can be classified so that they focus on the above and other issues. Our approach here was to take the suggestions of Brown and Walters (1983) and Mason (2000). Brown and Walters (1983: 1) asserts “that coming to know... is to commit ourselves to... operate on... the things we are trying to understand.” Important for the developed scheme is that questions developed for learners should be such that they work with the things that their responses indicate that they do not understand.

Regarding common errors Mason (2000) offers the strategy to let learners work with examples of such errors as exposed in their work and giving them opportunity to evaluate such incorrect ways of dealing with mathematical ideas. A further strategy offered by Mason (2000: 13) is to “Ask students to make up (and do) their own questions”. This strategy can be expanded by learners setting questions for other learners and that those who set the questions (and the answers), assess the answers of those other learners.

In fact in an in-service course for mathematics teachers in the 1980's, a teacher related how she used this strategy for setting and marking class tests. Her approach was to divide a class in groups of 5 to 6 learners. For, say, the first test group 1 would participate with her in setting and marking the class test. This procedure was followed for subsequent tests with other groups.

In response on whether learners would not leak the tests, she responded that learners were reluctant to do this since they were not certain on whether a next group will reciprocate. Further, she inculcated a strong ethic of honesty in her classes with the requisite consequences of losing marks if there is evidence tests were leaked.

Lastly, it is important to note that our deliberations in workshops rendered that a scheme must include questions that are generally commonplace in the boundary objects such as textbooks and previous examination papers.

The scheme presented below had its origin in these considerations and is explicated next.

### **A SCHEME OF MATHEMATICAL QUESTION TYPES**

The scheme consists of five question types. These are explained and exemplified below.

#### **STANDARD/DIRECT QUESTIONS:**

These are items that are normally used as examples in teaching, which generally appear in textbooks, previous examination and other learning resource materials and with which learners have a reasonable amount of practice. Their form might vary from what was used in teaching but these variations are small. These types of questions are normally effective for practising/testing factual or procedural knowledge.



*Examples:*

- (1) Simplify:  $2(2 - 3) - 5$
- (2) Solve form:  $3m - 7 - 5m = 5$
- (3) Complete the following: A kite is a quadrilateral with ...
- (4) Find the roots of  $x^2 - 2x - 7 = 0$  and state what the nature of the roots are.

### NON-STANDARD QUESTIONS TO CONVERT TO STANDARD FORM QUESTIONS:

The question (or part of it) is in a form that must be converted into some standard form to work towards its solution. These types of questions are especially useful for deepening knowledge and understanding of mathematics. They demand a higher level of thinking as the pathway to the solution is not directly implied in the way that the question is formulated.

*Examples:*

- (1) Find the HCF of  $2^2 \times 3 \times 5$  and 42
- (2) Solve for x:  $(x - 2)^2 = (x - 2) + 2$
- (3) Simplify completely: 
$$\frac{\sin(90^\circ + \theta) + \cos(180^\circ + \theta)\sin(-\theta)}{\sin 180^\circ - \frac{\sin 135^\circ}{\cos 135^\circ}}$$

### REVERSAL QUESTIONS:

The question is in the reverse form of the standard/direct question. In the quest for developing creativity and problem solving skills, these types of questions may be effectively used. Hence conceptual knowledge plays an important part in this type of questioning.

*Examples:*

- (1) A problem dealing with the simplification of an expression with integers which had three different operations gave -2 as the answer. Write down the problem. Is there only one answer?
- (2) Find a quadratic equation of which one root is irrational and the other a negative integer.
- (3) A function  $f(x)$  was differentiated and gave the answer:  $3x^5 + \frac{1}{2\sqrt{x}}$ . Find  $f(x)$ .



**EVALUATIVE QUESTIONS:**

These are questions which require learners to assess the correctness or not of produced answers. The most basic kind of this type of question is the true/false type. They include alternate correct ways of working and incorrect ways of working. A very good source for these kinds of questions is responses to questions in tests and examinations. This type of questioning develops meta-cognitive knowledge, in other words how one is thinking about mathematics and its procedures.

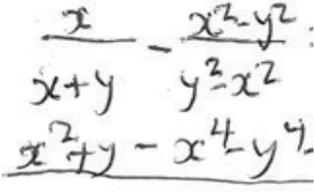
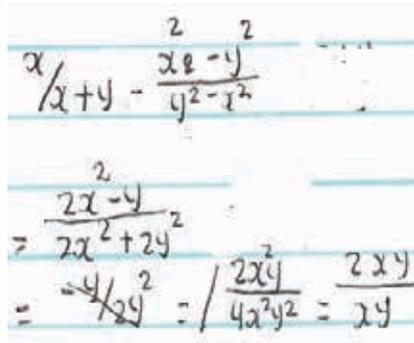
*Examples:*

(1) To simplify  $2(2 - 3) - 5$ , two learners did it as follows:

Learner A	Learner B
$2(2 - 3) - 5$ $= 2(2 - 3) - 2 - 3$ $= 2(2 - 3 - 1) - 3$ $= 2(-2) - 3$ $= -7$	$2(2 - 3) - 5$ $= 2(2 - 5) - 5 + 4$ $= 4 - 10 - 5 + 4$ $= 8 - 15$ $= -7$

Are their answers correct and did they use correct methods?

(2) To simplify  $\frac{x}{x+y} - \frac{x^2-y^2}{y^2-x^2}$  learners A and B worked as follows:

Learner A	Learner B
	

Do you agree with their way of working and answers?

- (3) For the question: Simplify  $\frac{\sin 104^\circ(2\cos^2 15^\circ - 1)}{\tan 38^\circ \cdot \sin^2 412^\circ}$  without the use of a calculator, a learner produced the following:  
Indicate with reasons where the learner went wrong.

$$\begin{aligned} & \frac{\sin 104^\circ (2 \cos^2 15^\circ - 1)}{\tan 38^\circ \cdot \sin^2 412} \\ & \approx \frac{\sin 104^\circ (2 \cos^2 15^\circ)}{\tan (90^\circ - 52^\circ) \cdot \sin^2 (360^\circ + 52^\circ)} \\ & = \frac{\sin (90^\circ + 14^\circ) \cdot 2 \cos^2 14^\circ}{\tan (90^\circ - 52^\circ) \cdot \sin^2 (360^\circ + 52^\circ)} \\ & = \frac{\sin 14^\circ \cdot 2 \cos^2 14^\circ}{\tan 52^\circ \cdot \sin^2 52^\circ} \\ & = \frac{\sin 14^\circ \cdot 2 \cos^2 14^\circ}{\tan 52^\circ \cdot \cos^2 \sin^2 52^\circ} \\ & = \frac{\sin 14^\circ \cdot 2 \cos^2 14^\circ}{\tan 52^\circ \cdot \sin^2 52^\circ} \end{aligned}$$

### LEARNER CONSTRUCTED QUESTIONS:

Although not strictly a question type, learners can be asked to develop questions (and their solutions) for their peers.

*Generic example:*

Develop a problem (and its solution) on (a topic, mathematical construct, etc.) for the rest of the class to solve.

### HOW DOES THIS SCHEME COMPARE WITH OTHER TYPOLOGIES

This question type scheme dovetails very well with the four types of knowledge as described by Krathwohl (2002, p214):

1. *Factual Knowledge (FK)*: knowledge of facts, properties, definitions, theorems, etc. accepted by the mathematics community
2. *Procedural Knowledge (PK)*: knowledge of algorithms and accepted ways of doing mathematics, e.g. solving equations.
3. *Conceptual Knowledge (CK)*: Knowledge of concepts e.g. similarity
4. *Meta-cognitive Knowledge (MCK)*: Knowledge of general problem solving strategies.



The table below illustrates this alignment between the types of questions and the types of knowledge.

	Types Knowledge	Question type
1	Factual	Direct questions
2	Procedural	Direct questions
3	Conceptual	Non-standard form questions; Reversal questions
4	Metacognitive	Evaluative questions; Learner constructed questions

We, however, are of the opinion that the presented scheme due to its focus on the surface level features of the questions are more easily appropriable by teachers for use in both their teaching and the development of school-based assessments. This is illustrated in the next section.

### EMERGING RESULTS OF USE OF THE SCHEME BY TEACHERS

The scheme was work-shopped in the first quarter 2014 with teachers participating in a continuing professional development initiative. As part of this workshop, the teachers had to develop questions according to the scheme for work they will do during the first quarter. In addition to standard/direct questions, teachers also generated questions of the other types.

Examples are:

*Learner-generated questions:*

Write a question on surds that requires the simplification of surds.

Write two questions of different difficulty level on rounding of decimal numbers.

*Evaluative questions:*

The following question was in a test: Simplify:  $\frac{(5a)^{-2}}{5a^{-3}}$

Three learners gave the following answers

*Learner 1: a*

*Learner 2: 5a*

*Learner 3:  $\frac{a}{125}$*

Which learner is correct? Show your calculations and justify your answer.

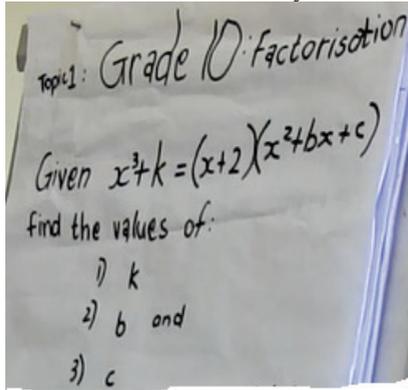
Explain the mistakes made by the other two learners.

Is  $\sqrt{19}$  rounded off to one significant number equal to 4.

*Non-standard form questions:*

If  $A = 2^n$  show that  $\frac{4^{n+2} \cdot 9^{n-1}}{72^n \cdot 2^{1-n}} = 0,89$ .

At a later workshop in the quarter teachers reported back on their use of question types in the classroom. A teacher reported that she gave her learners the question in the figure below after dealing with the factorisation of two cubes. She explained how she solved the problem and after some discussion on her way of working and possible alternatives, one of the facilitators asked: “how do you teach such problems?”



The teacher responded:

After I did the sum and difference of cubes...I mean one of the exercises in the textbook has a lot of questions on the sum and difference of cubes. So what I did was, I went through each one and told them to analyse it. And...once we've factorize it, I told them look at...in your short bracket, look at your first term and in your long bracket, which was that one [points to the bracket containing the trinomial in the figure above] look at the first term of the long bracket and what is the relationship in each and every one of them. And then they said “Oh...That is that [pointing to  $x^2$ ] is that one [pointing  $x$ ] squared.” So I said that is what it's gonna be in every single case. And then I told them “Look at the middle term [whilst pointing to  $bx$ ] and try to find what is the relationship between these two numbers [shifting the pointing to the 2 in the first bracket]? They went through and they said “OK. It's the same number with the same digits but the signs are different” [some faint interjection from another participant {additive inverse;}. So I told them “That is what it is gonna be in every single case. That is what they did'nt know, that it's called additive inverse so I just told them. They have to therefore [inaudible]”. And I told them to look at this one [points to the 2 in the 'small' bracket] and this one [points to the  $c$  in the 'long' bracket] and then they said “Oh, that's the square again.” Then I told them to look at what they starting with [points to  $k$  on the right-hand side] and look at the first bracket [pointing to the 2] and they said “Ok, Miss, it like that thingie with the three on it for the first number and the second number.” So that is how I taught it to them.



What the above report-back of a teacher illustrates is that some teachers are beginning to appropriate the scheme and using it, albeit in a limited instances, in their practice. The above is an instance of a reversal question, other teachers reported on using evaluative questions in their practice.

## CONCLUSION

Schemes for classifying question types are always contestable and context-bound. What might, for example, be a reversal question for one might be a standard/direct question for another. Teachers with their intimate knowledge of their learners, their own practice and various issues that contribute towards the determination and enactment of their goals know best which question types can be classified as which for their learners.

We, however, contend that classification schemes that emerge from their concerns and dilemmas have great currency for inspiring them to extend the repertoire of question types learners are normally confronted with. This, we believe, will open ways for more productive teaching and ultimately more meaningful learning.

## REFERENCES:

- Anderson, L.W. & Krathwohl D.R. (Eds). (2001) A taxonomy for learning, teaching and assessing: A revision of Bloom's taxonomy of educational objectives. New York: Longman.
- Boaler, J. & Brodie, K. (2004). The importance, nature and impact of teacher questions. In D. E. McDougall & J. A. Ross (Eds.), Proceedings of the 26th Annual meeting of the North American International Group for the Psychology of Mathematics Education. pp.773–781. Toronto, Japan: University of Toronto.
- Brown, S. I. & Walter, M. I. (1983). The art of problem solving. Philadelphia, Pennsylvania: The Franklin Institute Press.
- Department of Basic Education. (2013). Report on the 2013 NATIONAL Senior Certificate. Pretoria
- Julie, C. (2012). The primacy of teaching procedures in school mathematics. In Nieuwoudt, S, Laubser, D. & Dreyer, H. (Eds.), Proceedings of the 18th Annual National Congress of the Association for Mathematics Association of South African. Potchefstroom: North-West University.
- Julie, C. (2013). Towards a model for intentional teaching for improving achievement in high-stakes mathematics examinations. In Z. Davis & S. Jaffer (Eds.), Proceedings of the 19th Annual Congress of the Association for Mathematics Education of South Africa, Vol. 1. (pp. 86 – 96). Cape Town: AMESA.
- Kilpatrick, J. (1987). Problem formulating: Where do good problems come from. Cognitive science and mathematics education, 123-147.
- Krathwohl, D. (2002). A revision of Bloom's taxonomy: an overview. Theory into practice. 41(4)
- Kutcheman, D. (eds) (2007). Proceedings of the British Society for Research into Learning Mathematics, 27(3)
- Schoenfeld, A. H. (Ed.). (1987). Cognitive science and mathematics education. Psychology Press.
- Mason, J. (2000). Practice guide: Teaching Mathematics. Milton Keynes: The Open University.
- Tanner, H. & Jones, S. (2003). Self-Efficacy In Mathematics And Students' Use Of Self- Regulated Learning Strategies During Assessment Events. International Group for the Psychology of Mathematics Education.

- Watson and De Geest (2012). Learning coherent mathematics through sequences of microtasks: Making a difference for secondary learners. *International Journal of Science and Mathematics Education*, 10: 213 – 235
- Wright, L. (2013) Towards ‘Discourse 4’: Re-orientating research discourse to address a key aspect of South Africa’s education crisis. *Southern African Review of Education*,



# A COMPARISON OF THE ACHIEVEMENT GOAL ORIENTATION OF MATHEMATICS LEARNERS WITH/WITHOUT ATTENTION DEFICIT HYPERACTIVITY DISORDER (ADHD)

Erica D. Spangenberg

University of Johannesburg

*This paper reports on the achievement goal orientation of Grades 8-9 mathematics learners with/without ADHD in the Ekurhuleni-East District in South Africa. The research question is: How does the achievement goal orientation of learners suffering from ADHD and those without ADHD compare? A social constructivism paradigm with a quantitative design was adopted. Ten learners suffering from ADHD and 10 learners without ADHD completed structured questionnaires. The results revealed that non-ADHD learners perceived peers' mastery goals orientation and behavioral and cognitive engagement significantly higher; while ADHD learners regarded personal performance avoid goals orientation, perception of parents' mastery emphasis and their perception of parents' performance emphasis significantly higher.*

## FOCUS

The purpose of this paper is to compare the achievement goal orientation of Grades 8-9 mathematics learners suffering from attention deficit hyperactivity disorder (ADHD) with those learners not suffering from ADHD in a school in Ekurhuleni-East. Achievement goal orientation is based on a modern 'goal-as-motives' theory suggesting that 'all actions are given meaning, direction and purpose by the goals that individuals seek out, and that the quality and intensity of behaviour will change as these goals change' (Covington, 2000, p. 174). By reinforcing particular goals and disregarding others, a teacher can influence and ultimately change the way in which learners learn and thus change their motivation (Covington, 2000), which could lead to better performance in mathematics. Consequently, the question is: What is the achievement goal orientation of learners suffering from ADHD and those not suffering from ADHD?

## THEORETICAL FRAMEWORK AND LITERATURE OVERVIEW

Learners' goals influence their learning activities, engagement in tasks, attitude towards learning and consequently their achievement. According to Vedder-Weiss and Fortus (2011, p.200) the adoption of different goal orientations lead to differences in the way learners engage with school work and their consequent emotional experiences at school.

DuPaul, Volpe, Jitendra, Lutz, Lorah and Gruber (2004) examined different factors, including behavioural observations that determine academic achievement in the ADHD learners. In particular, the strongest factor for academic achievement found was teachers' perceptions of academic skills. Thus, interventions should not only include the use of medications, but also be combined with academic support.

According to Zentall (1990) Intelligence Quota (IQ) and reading comprehension skills showed no indications concerning ADHD learners' mathematical skills. However, calculation speed and task behaviour had significant importance pertaining to their performance and goal achievement. On the contrary, Lamminmaeki, Ahonen, Naerhi, Lyytinem and Todd de Barra (1995) found that learners with ADHD were no more impaired in mathematics than others without ADHD.

Zentall, Smith, Lee and Wieczorek (1994) discovered that the ADHD boys all showed a lower problem-solving ability. The poor performance and achievement goal orientation was accredited to the various subtypes of ADHD found typically as a behavioural symptom: distractibility, hyperactivity and impulsivity. Zentall et al. (1994) believed that when an ADHD child is distracted, the child is attempting to lessen their under-stimulated minds by seeking task or reactions that would increase the levels of stimulation. Thus, by using external stimulating factors during a boring but routine task, the child with ADHD will perform better.

Given the limited information and research on mathematical understanding and calculations within ADHD learners, very few conclusions can be made. However, it has been shown that children with ADHD are slower and less accurate when conducting calculations than those non-ADHD children. Thus, there is conjecture that the over-working of memory causes the need to solve and calculate which then causes the deficit in the achievement goal orientation and performance. Moreover, the poor achievement goal orientations in calculations and solving in mathematics may be associated with hyperactivity and distractibility, two major indicators of ADHD.



Much research (Ackerman, Dyckman & Oglesby, 1983; Zentall, 1990; Zentall & Ferkis, 1993; Zentall, Smith, Lee & Wieczorek, 1994; Lamminmaeki, Ahonen, Naerhi, Lyytinen & Todd de Barra, 1995; Marshall, Hynd, Handwerk & Hall, 1997) has been dedicated to the negative impact of ADHD on school performance. However, most research on academic success and ADHD has focussed on reading disorders in children with ADHD rather than difficulties in mathematics (Lucangeli & Cabrele, 2006, p.53).

August and Garfinkel (1990) examined the reading ability of ADHD boys', diagnosed in a university outpatient clinic, against the reading ability of non-ADHD boys. Barry, Lyman and Klinger (2002) focussed on the negative consequences that ADHD has on an individual's academic achievements due to behaviour and argued that:

children with ADHD experience shortfalls in some of the abilities establishing the executive functions such as planning, organising, maintaining an appropriate problem-solving set to achieve a future goal, inhibiting an inappropriate response or deferring a response to a more appropriate time representing a task mentally (i.e. in working memory), cognitive flexibility and deduction based on limited information. (p.274)

A study that identifies the achievement goal orientation of Grades 8-9 mathematics learners suffering from ADHD in relationship to those learners not suffering from ADHD is new to South Africa, even though there have been investigations into the types of goals learners assume in the classroom and the contextual factors which play a role in learners' choices of goals and learning activities (Tapola & Niemivirta, 2008; Vedder-Weiss & Fortus, 2011). In particular, Vedder-Weiss and Fortus (2011) referred to a wide range of motivation research and conclude that declines in motivation and attitude toward learning have been common across learning areas and are often linked to changes in classroom environment.

## RESEARCH METHODOLOGY

### Research design

Vandeyar (2010, p.87) noted that social investigations are bound in the 'consideration of how certain phenomena or forms of knowledge are achieved by people in action', which convinced me to adopt a social constructivism theoretical paradigm. Moreover, this paradigm is tailored to an investigation of how learners, parents and teachers perceive achievement goal orientation of the school and classroom environment and the way in which these perceptions inform and shape their choice of a specific goal orientation.

I utilised a quantitative technique in the study (Creswell & Clark, 2006). I established learners' perceptions regarding achievement goal orientation quantitatively through questionnaires. In order to compare the achievement goal orientation of learners suffering with ADHD and those not suffering from ADHD, the following hypothesis was interrogated in the quantitative approach:

There are significant differences in the achievement goal orientation of learners suffering from ADHD and those not suffering from ADHD.

## Sample

A purposeful sampling technique (Creswell, 2003) was used to select grades 8-9 mathematics learners suffering from ADHD and those not suffering from ADHD from one secondary school in a single district in South Africa, namely the Ekurhuleni-East District. Criterion sampling was utilised (Palys, 2008). The area was chosen for I had easy access to the school and the participants were selected via postings in the school's weekly newsletter, through private discussions with the school counsellor and through parent evening discussions with the parents of those learners, suffering from ADHD. Participation was voluntary, consent was obtained and the anonymity of the participants was protected (Mouton, 2001). From a population of 540 grades 8-9 learners in the school, 10 non-ADHD learners and 10 ADHD learners participated voluntarily. Sixteen of the 20 learners were in one class. Furthermore, all the learners in the study were proficient in reading, speaking and writing in English. Moreover, I could only utilise a natural formed group, namely learners in a classroom setup for this research, which justifies a convenience sample (Creswell, 2003). Participants in the ADHD sample were required to have a diagnosis of ADHD from a physician or psychologist, but no diagnosis of a neurological disorder or genetic syndrome, for example pervasive developmental disorders, psychotic disorders or Tourette's disorder. Also, the learners without ADHD were required not to have any previous diagnosis of ADHD or any learning or behaviour problems identified by parents. Seven of the learners suffering from ADHD, were taking psycho-stimulant medication for their symptoms, for example Ritalin or Concerta. However, on the data gathering date, participants were asked to be medication-free. Eight learners suffering from ADHD were receiving some form of special education service, including support from an educational tutor.

## Data collection: Questionnaires

The quantitative feature was a structured questionnaire based on an existing standardised instrument. Permission was obtained to utilise and amend a questionnaire for mathematics, developed by Veder-Weiss and Fortus (2011) for a similar study in Israel comparing grades five to eight learners' goal orientations in science learning. The questionnaire consisted of 89 mixed survey items with a 1-5 point Likert scale (1 = Not true at all and 5 = Very true) relating to 17 motivation



constructs. The questionnaires were completed in test conditions and took approximately one hour.

The reasons for using the questionnaire were firstly, the questionnaire was standardised, so it allowed minimal misinterpretation to occur concerning the information presented to them. This particular issue was solved by piloting the questions. The pilot group consisted of two individuals, whom were not participating in the final research project.

One individual was clinically diagnosed with ADHD and the second individual was non-ADHD. Secondly, using a questionnaire is relatively a quick way to collect information.

### **Data analyses**

The results for each question in the questionnaire were calculated and categorised into 17 key motivation constructs. Thereafter, the Mann-Whitney U Test, as appropriate non-parametric statistical technique, was undertaken to examine differences between the medians of the responses of non-ADHD learners and ADHD learners on the 17 key motivation constructs respectively.



### Reliability

The internal consistency of each of the of the 17 key motivation constructs was determined by using the Cronbach  $\alpha$  coefficient as presented in Table 1. A score of 0.7 and higher was assumed as reliable.

Construct	No. of items	Cronbach $\alpha$
Learners' perception of teacher's mastery goals emphasis	8	0.683
Learners' perception of teacher's performance approach goals emphasis	4	0.746
Learners' perception of teacher's performance avoid goals emphasis	4	0.701
Learners' perception of school's mastery goals emphasis	5	0.694
Learners' perception of school's performance goals emphasis	5	0.722
Learners' personal mastery goals orientation	7	0.715
Learners' personal performance approach goals orientation	5	0.675
Learners' personal performance avoid goals orientation	5	0.677
Learners' self-efficacy	5	0.716
Learners' perception of peers' mastery goals orientation	4	0.709
Learners' perception of peers' performance approach goals orientation	4	0.693
Learners' perception of peers' performance avoid goals orientation	4	0.678
Learners' perception of parents' mastery emphasis	5	0.717
Learners' perception of parents' performance emphasis	4	0.727
Behavioral and cognitive engagement	5	0.716
Active extra-curricular engagement	7	0.763
Active extra-curricular rejection	6	0.761

Table 1: Internal reliability of the of the motivation constructs.



**Validity**

Veder-Weiss and Fortus (2011) granted permission for the amendment and usage of their questionnaire on goal orientations in science learning and the intellectual property rights of it were recognised. The questionnaire had already complied with all validity aspects. To ensure face and content validity, the questionnaire was shown to colleagues for comments and inputs, to ensure that the constructs were clearly conceptualised. Consequently, the questionnaires were amended with regard to timeframes, terminology, readability and clarity. The purpose was to ensure coherency and consistency of the questions. The questionnaires were administered under examination conditions. All the participants’ contributions, including literature, were recognised by proper referencing.

**MAIN FINDINGS**

**Findings from questionnaires**

The Mann-Whitney U Test was undertaken to examine differences between the medians of the responses of non-ADHD learners and ADHD learners on the 17 key motivation constructs respectively. Table 2 presents data on the calculated z-values and the approximately calculated statistical significance of differences between the crossed variables. A correlation at the 0.05 level was assumed as significant.

Constructs	Mann-Whitney U	Wilcoxon W	Z	Asymp. Sig. (2-tailed)	Exact Sig. (2-tailed)	Exact Sig. (1-tailed)
Learners' perception of teacher's mastery goals emphasis	34.000	89.000	-1.218	.223	.247	.124
Learners' perception of teacher's performance approach goals emphasis	30.500	85.500	-1.492	.136	.143	.072
Learners' perception of teacher's performance avoid goals emphasis	46.000	101.000	-.303	.762	.796	.400
Learners' perception of school's mastery goals emphasis	47.000	102.000	-.229	.819	.853	.427
Learners' perception of school's 0.481performance goals 0.029emphasis	47.000	102.000	-.229	.819	.853	.427

Learners' personal mastery goals orientation	32.500	87.500	-1.334	.182	.190	.100
Learners' personal performance approach goals orientation	40.000	95.000	-.760	.447	.481	.024
Learners' personal performance avoid goals orientation	21.000	76.000	-2.209	.027*	.029*	.015
Learners' self-efficacy	43.500	98.500	-.498	.619	.631	.316
Learners' perception of peers' mastery goals orientation	22.500	77.500	-2.127	.033*	.035*	.018
Learners' perception of peers' performance approach goals orientation	34.000	89.000	-1.220	.222	.247	.124
Learners' perception of peers' performance avoid goals orientation	28.500	83.500	-1.645	.100	.105	.053
Learners' perception of parents' mastery emphasis	17.500	72.500	-2.533	.011*	.11*	.056
Learners' perception of parents' performance emphasis	13.000	68.000	-2.824	.005*	.004*	.001
Behavioral and cognitive engagement	12.500	67.500	-2.866	.004*	.003*	.001
Active extra-curricular engagement	48.500	103.500	-.114	.909	.912	.456
Active extra-curricular rejection	42.000	97.000	-.607	.544	.579	.290

\*Correlation is significant at the 95% level

Table 2: Test statistics of learners without/with ADHD and the motivation constructs.



Learners with/without ADHD differed significantly at a 95% level in terms of five of the 17 motivation constructs mentioned by Veder-Weiss and Fortus (2011), namely learners' personal performance avoid goals orientation ( $p = .027 < .05$ ); learners' perception of peers' mastery goals orientation ( $p = 0.033 < 0.05$ ); learners' perception of parents' mastery emphasis ( $p = .011 < .05$ ); learners' perception of parents' performance emphasis ( $p = .005 < .05$ ); and behavioural and cognitive engagement ( $p = .004 < .05$ ).

As there were statistical significant differences between crossed variables, there was a need to analyse the data shown in table 3, indicating which continuous variable was higher on average.

Construct	Independent variables	N	Mean Rank	Sum of Ranks
Learners' perception of teacher's mastery goals emphasis	Non-ADHD	10	12.10	121.00
	ADHD	10	8.90	89.00
Learners' perception of teacher's performance approach goals emphasis	Non-ADHD	10	8.55	85.50
	ADHD	10	12.45	124.50
Learners' perception of teacher's performance avoid goals emphasis	Non-ADHD	10	10.90	109.00
	ADHD	10	10.10	101.00
Learners' perception of school's mastery goals emphasis	Non-ADHD	10	10.20	102.00
	ADHD	10	10.80	108.00
Learners' perception of school's performance goals emphasis	Non-ADHD	10	10.20	102.00
	ADHD	10	10.80	108.00
Learners' personal mastery goals orientation	Non-ADHD	10	12.25	122.50
	ADHD	10	8.75	87.50
Learners' personal performance approach goals orientation	Non-ADHD	10	9.50	95.00
	ADHD	10	11.50	115.00
Learners' personal performance avoid goals orientation	Non-ADHD	10	7.60	76.00
	ADHD	10	13.40	134.00
Learners' self-efficacy	Non-ADHD	10	11.15	111.50
	ADHD	10	9.85	98.50
Learners' perception of peers' mastery goals orientation	Non-ADHD	10	13.25	132.50
	ADHD	10	7.75	77.50

Learners' perception of peers' performance approach goals orientation	Non-ADHD	10	8.90	89.00
	ADHD	10	12.10	121.00
Learners' perception of peers' performance avoid goals orientation	Non-ADHD	10	8.35	83.50
	ADHD	10	12.65	126.50
Learners' perception of parents' mastery emphasis	Non-ADHD	10	13.75	137.50
	ADHD	10	7.25	72.50
Learners' perception of parents' performance emphasis	Non-ADHD	10	6.80	68.00
	ADHD	10	14.20	142.00
Behavioral and cognitive engagement	Non-ADHD	10	14.25	142.50
	ADHD	10	6.75	67.50
Active extra-curricular engagement	Non-ADHD	10	10.65	106.50
	ADHD	10	10.35	103.50
Active extra-curricular rejection	Non-ADHD	10	9.70	97.00
	ADHD	10	11.30	113.00

Table 3: Motivation constructs of non-ADHD learners and ADHD learners.

From this data, it can be concluded that, non-ADHD learners regarded their goal orientation significantly higher than ADHD learners, pertaining to:

- Learners' perception of peers' mastery goals orientation (non-ADHD  $Mdn = 5$  vs. ADHD  $Mdn = 4$ ),  $U = 34.0$ ,  $p = .033 < .05$  (at the 95% level of confidence),  $r = .27$  (a finding with a low to moderate practical significance);
- Behavioral and cognitive engagement (non-ADHD  $Mdn = 5$  vs. ADHD  $Mdn = 4$ ),  $U = 12.5$ ,  $p = .004 < .05$  (at the 95% level of confidence),  $r = .03$  (a finding with a low practical significance);

In contrast, ADHD learners regarded their goal orientation significantly higher than non-ADHD learners, pertaining to:

- Learners' personal performance avoid goals orientation (non-ADHD  $Mdn = 3$  vs. ADHD  $Mdn = 4$ ),  $U = 21.0$ ,  $p = .027 < .05$  (at the 95% level of confidence),  $r = .49$  (a finding with moderate to high practical significance);
- Learners' perception of parents' mastery emphasis (non-ADHD  $Mdn = 5$  vs. ADHD  $Mdn = 4$ ),  $U = 17.5$ ,  $p = .011 < .05$  (at the 95% level of confidence),  $r = .57$  (a finding with moderate to high practical significance);
- Learners' perception of parents' performance emphasis (non-ADHD  $Mdn = 4$ , vs. ADHD  $Mdn = 5$ ),  $U = 13.0$ ,  $p = .005 < .05$  (at the 95% level of confidence),  $r = .63$  (a finding with moderate to high practical significance);



The above-mentioned findings supported the hypothesis about the comparisons between five key motivation constructs and the non-ADHD and ADHD learners.

### **Discussion and conclusion**

The paper focused on comparing the achievement goal orientation of Grades 8-9 mathematics learners suffering from attention deficit hyperactivity disorder (ADHD) with those learners not suffering from ADHD in a school in Ekurhuleni-East.

The results from the quantitative data address the research question, namely that non-ADHD learners regarded their goal orientation significantly higher than ADHD learners, pertaining to their perception of peers' mastery goals orientation and behavioural and cognitive engagement. On the other hand, ADHD learners regarded their goal orientation significantly higher than non-ADHD learners, pertaining to their personal performance avoid goals orientation, perception of parents' mastery emphasis and their perception of parents' performance emphasis. Thus, ADHD learners, like the parents, as noted by Vedder-Weiss and Fortus (2013) perceive goals that parents emphasise better predictors of their motivation, than perceptions of the goals that peers emphasise.

Limitations to the study were that the sample was small. This research was aimed at determining the achievement goal emphasis of ADHD and non-ADHD learners at one particular school, in a single district in a single province in South Africa. As a result, there is a low external validity, as the study conducted cannot be generalised to other situations. Given the localised nature of this study, I recommend that the results obtained be confirmed through similar studies of this nature and in other provinces in South Africa. In this way, a better understanding of the environmental factors that affect non-ADHD and ADHD learners' motivation in Mathematics can be obtained.

Rather than focusing on the difficulties that learners have in Mathematics, most of the research on academic success and ADHD has focused on reading disorders. Further research is needed into areas of education concerning ADHD and mathematics understanding. The ways in which empirical realities manifest are much more complex than the broad groupings pointed to in the literature in this paper. Hence, I suggest that further research in this regard should be conducted.

In conclusion, learners' achievement goal orientations can be taken into consideration, when teachers plan learning activities and engage learners in tasks, which can ultimately influence learners' attitudes towards learning and consequently their achievement. One avenue available to teachers to ensure this sustained interest and involvement in quality learning is to explore individual learner's achievement goal orientations and to examine those factors which play a role in and are responsible for developing these goal orientations. Consequently, teachers can then cultivate and promote appropriate goal orientations amongst their learners which lead to an improvement in academic achievement.

## REFERENCES

- Ackerman, P. T., Dyckman, R. A., & Oglesby, D. M. (1983). Sex and group differences in reading and attention disordered children with and without hyperkinesis. *Journal of Learning Disabilities*, 16, 407-415.
- August, G. J., & Garfinkel, B. D. (1990). Comorbidity of ADHD and reading disability among clinic referred children. *Journal of Abnormal Child Psychology*, 18 (1), 29-45.
- Barry, T. D., Lyman, R. D., & Klinger, L. G. (2002). Academic underachievement and attention deficit/hyperactivity disorder: The negative impact of symptom severity on school performance. *Journal of School Psychology*, 40, 259-283.
- Creswell, J.W. (2003). *Research design: Qualitative, quantitative and mixed methods approaches*. (2nd ed.). Thousand Oaks, CA: Sage.
- Creswell, J.W., & Plano Clark, V.L. (2006). *Designing and Conducting Mixed Methods Research*. America: Sage Publications.
- Covington, M. V. (2000). Goal theory, motivation and school achievement: An integrative review. *Annual Review of Psychology*, 51, 171-200.
- DuPaul, G. J., Volpe, R. J., Jitendra, A. K., Lutz, J. G., Lorah, K. S., & Gruber, R. (2004).
- Lamminmaeki, T., Ahonen, T., Naerhi, V., Lyytinem, H., & Todd de Barra, H. (1995). Attention deficit hyperactivity disorder subtypes: Are there differences in academic problems? *Developmental Neuropsychology*, 11, 297-310.
- Lucangeli, D., & Cabrele, S. (2006). Mathematical difficulties and ADHD. *Exceptionality*, 14 (1), 53-62.
- Marshall, R. M., Hynd, G. W., Handwerk, M. J., & Hall, J. (1997). Academic underachievement in ADHD subtypes. *Journal of Learning Disabilities*, 30, 635-642.
- Mouton, J. (2001). *How to succeed in your master's & doctoral studies*. Pretoria: Van Schaik.
- Palys, T. (2008). Purposive Sampling. In L. M. Given (Ed.). *The Sage Encyclopedia of Qualitative Research Methods*. Vol 2 (pp.697-698). Thousand Oaks, California: Sage.
- Schweitzer, J.B., Cummins, T.K., & Carole A. Kant, C.A. (2001). Attention-Deficit/Hyperactivity Disorder. *Medical Clinics of North America*, 85(3), 757-777.
- Tapola, A., & Niemivirta, M. (2008). The role of achievement goal orientations in students' perceptions of and preferences for classroom environment. *British Journal of Psychology*, 78, 291-312.
- Vandeyar, T. (2010). *The appearance of education policy on information and communication technology in South African Schools*. Unpublished doctoral dissertation, University of Pretoria, Pretoria, South Africa.
- Vedder-Weiss, D., & Fortus, D. (2011). Adolescents' Declining Motivation to Learn Science: Inevitable or Not? *Journal of Research in Science Teaching*, 48(2), 199–216. <http://dx.doi.org/10.1002/tea.20398>
- Vedder-Weiss, D., & Fortus, D. (2013). School, teacher, peers, and parents' goals emphases and adolescents' motivation to learn science in and out of school. *Journal of Research in Science Teaching*, 50(8), 952–988. <http://dx.doi.org/10.1002/tea.21103>
- Zentall, S. S. (1990). Fact-retrieval automatization and math problem-solving: Learning disabled, attention disordered and normal adolescents. *Journal of Educational Psychology*, 82, 856-865.
- Zentall, S. S., & Ferkis, M. A. (1993). Mathematical problem solving for youth with ADHD, with and without learning disabilities. *Learning Disabilities Quarterly*, 16, 6-18.
- Zentall, S. S., Smith, Y. N., Lee, Y. B., & Wieczorek, C. (1994). Mathematical outcomes of attention-deficit hyperactivity disorder. *Journal of Learning Disabilities*, 27, 510-519.



# PEDAGOGIC ACTIONS AND STRATEGIES THAT SUPPORT SOPHISTICATED WITHIN FOUNDATION PHASE

## NUMBER WORK

Marie Weitz

Wits School of Education

### INTRODUCTION

A number of analysts describe the low performance of South African students in mathematics as ‘a crisis’ (Schollar, 2009; Fleisch, 2007; Van der Berg and Louw, 2006). Van der Berg and Louw (2006) emphasize South Africa’s poor scores in mathematics tests compared to other countries in Africa. They also argue that tests scores obtained by South African students on international tests are much lower than those obtained by students of the same age in France, Hong Kong, Singapore and South Korea (Van der Berg & Louw, 2006). Ensor et al. (2009) and Fleisch (2007) note that international studies show South Africa’s performance for learners in numeracy is lower than that of eleven other African countries. Concerning the Foundation Phase more specifically, there is evidence of a lack of shift from concrete counting-based strategies to more abstract calculation-based strategies (Ensor et al., 2009) with Schollar (2009) indicating the prevalence of concrete counting strategies well into the Intermediate Phase.

Wright et al (2006) suggest that children need to move towards more sophisticated strategies within work on early number. Other researchers describe this shift in terms of supporting children to move to more abstract models and strategies within early number (Ensor et al., 2009; Fleisch, 2008; Schollar, 2009). In this paper, drawing from my Masters study and literature on early number teaching, linked with findings of early number teaching in South Africa, I present a summary of pedagogic actions and strategies that can be used with common resources to support progression.

### Concrete/abstract strategies

Concrete counting-based strategies refer to actions where the learner cannot find the answer to a mathematical problem without using concrete objects. This means that learners cannot solve problems without concrete counting, drawing tallies or using perceptual strategies such as feeling or seeing items. In contrast, abstract calculation-based strategies involve strategies where the child does not need concrete objects to find the answer, but can instead use mental calculations in which numbers have been transformed into abstract entities that can be operated on. Ensor et al. (2009) found that concrete methods to solve problems, such as tally counting, were dominant in the Western Cape schools they investigated through classroom-based research. They argued that these learners’ poor mathematical results were the result of insufficient moves from counting to calculating.

Sfard (1992), writing more theoretically, states that “the term abstracting is commonly used with reference to the activity of creating concepts that do not refer to tangible, concrete objects” (p. 111). A more practical explanation of abstraction is provided by Haylock & Cockburn (2008, p. 42): “when a child adds 3 and 4, what actual objects are, makes no difference to the mathematical process: 3 sweets and 4 sweets, 3 boys and 4 boys, 3 counters and 4 counters, these are all presented in the same abstraction  $3+4=7$ ”. A child who can transfer the ‘3 counters and 4 counters’ task to her fingers has a more abstract view of numbers than the child who can only count by seeing or feeling counters..

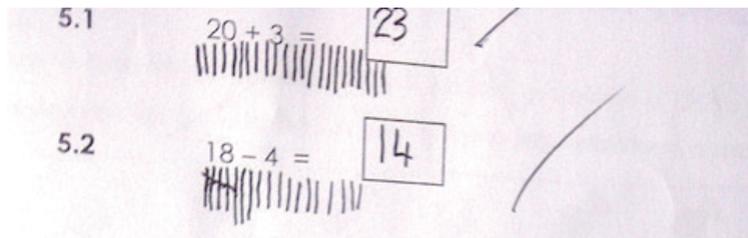
### Objectification

To have an abstract understanding of number, a child should be able to appreciate the quantities underlying numerical symbols without the need for counting. Sfard (2010) argues that *abstract objects* are outcomes of *reification*. The basic principle of reification is that the operational (process-orientated) conception emerges first and over time, becomes reified into the mathematical object (structural conception) (A Sfard & Linchevski, 1994). The *theory of reification* refers to the turning of processes into objects. Sfard (1991) argues: “It seems therefore, that the structural approach should be regarded as a more advanced stage of concept development. In other words, we have good reason to expect that in the process of concept formation, operational conceptions would precede the structural”. Sfard (1992) also draws attention to the fact that process and object are inseparable: they are different facets of the same thing. If we look at number sense in the way Sfard’s (1992) reification theory suggests, we can say that concrete counting is a way of understanding numbers as a process. When a learner understands number in a more abstract way, (i.e. the number exists without needing to be being associated with concrete counting actions), the learner has a structural understanding of numbers and reification has taken place. Sfard (2008) argues: “Such an act of reification - of discursively turning processes into objects - is the beginning of *objectification*, which, if completed, will leave us convinced about the mind-independent, ‘objective’ existence of the object-like referent” (p. 44). Sfard (2008) states that objectification is the process which involves “... two tightly related, but not inseparable discursive moves: *reification*, which consist in substituting talk about actions with talk about objects, and *alienation*, which consists in presenting phenomena in a impersonal way, as if they were occurring of themselves, without the participation of human beings” (p. 44). The author argues that objectification builds efficiency in mathematical communication because a learner/teacher can refer to mathematical objects and does not need to describe them in processes terms. Ensor et al. (2009) showed in their study that most of their learners still used concrete methods to find answers to mathematical problems in Grade 3. When children use concrete methods to find the answer, this shows that they have not yet objectified numbers, and remain reliant instead on counting processes.

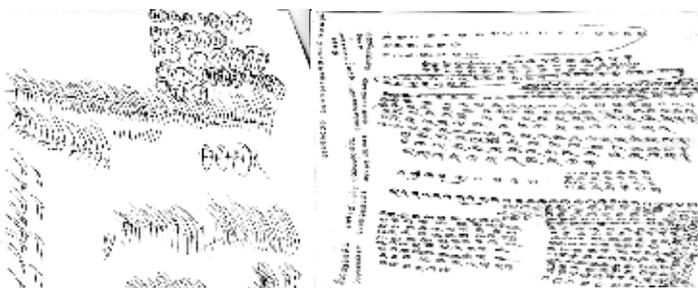


### Example of a process conception of number

In my Masters study, learner responses on the Grade 1 ANA indicated examples of process conceptions of number. A grade 2 child used tallies to find the answer to  $20+3=$  \_\_. His working, detailed below, shows that he made 20 tallies and then another 3 tallies and, counting these together, got the correct answer.



The question was allocated 1 mark for a correct answer, which he was awarded. In writing emanating from this study (Weitz & Venkat, 2013) we noted that mark allocations like these in the early grades are likely to work against persuading teachers to encourage the development of more sophisticated strategies within counting, and early addition and subtraction tasks. In the paper, we argued that a child who understands the ‘tens-based’ number structure and the meanings that can be associated with addition and the counting number sequence, would know that  $20+3$  should be 23 without drawing tallies. In this working, neither 20 nor 3 exist as an abstract concept – both numbers only exist as outcomes of concrete unit counting processes. We also noted that the low number range in the Grades 1 and 2 ANAs makes concrete methods feasible as means for evaluating answers to problems, but the increasing number ranges in the curriculum for Grade 3 and beyond, makes ongoing use of unit counting highly cumbersome and error-prone, as Schollar’s (2009) report shows:



While many researchers emphasize the need for the shift from concrete to abstract number conceptions, Wright et al. (2006) developed a framework where this shift is described in fine nuances, with stages of progression. Their model, developed from research done by Steffe and colleagues (for example: Steffe, 1992; Steffe and Cobb, 1988; Steffe et al., 1983), focuses particularly on developing children's strategies for solving number problems. Focusing centrally on counting, addition and subtraction, Wright et al. (2006) see counting as a developmental process which they break down into 6 stages: emergent; perceptual; figurative-counting; initial number sequence; intermediate number sequence; and facile number sequence. These stages are referred to as the Stages of Early Arithmetical Learning (SEAL). The details of each stage below underscore that the focus is not simply on whether the child can count, but also the strategies that the child uses to count.

These SEAL stages take the shift from concrete to abstract into account, and break it down to a more detailed set of developmental indicators, and are detailed below:

Perceptual Counting, (stage 1 of SEAL), refers to a child counting by seeing or feeling objects. This is a concrete way of dealing with number. A child on this stage cannot yet associate his fingers with the numbers 4 and 5 to find the answer to  $4+5$ ; instead concrete counting objects are required.

In Figurative Counting (stage 2), a child can count unseen objects, but starts at one when she is asked to add screened objects ('screened objects' are objects that are not visible to the learner). In this stage, the child has awareness of 4 and 5 as transferable to their fingers. At this stage she will count out 4 on one hand and then count out five on the other hand and then count them all from 1-9 keeping track with her fingers. Stages 1 and 2 use the 'Triple count method' (Anghileri, 2006)

At Stage 3 which is the Initial Number Sequence, a child uses counting-on strategies rather than count-all strategies. At this stage, the child can use 'count-down-from' strategies to solve removed items tasks, but cannot use 'count-down-to' strategies to solve missing subtrahend tasks.

In the Intermediate Number Sequence stage (stage 4), the child can additionally count down to as well as count down from and can choose the more efficient strategy for particular addition and subtraction problems. For a subtraction task like  $11-3$ , counting-down-from-11 and removing 3 is more efficient, but for  $11 - 7$  count-on-from-7 or count backwards to 7 is more efficient. If we set the following problem for  $5 - ? = 3$  (missing subtrahend) in a problem context, a child on stage 4 knows that she can count down from 5 and say 4, 3 ... and state the answer of 2. A child at stage 3 would not be able to answer this question, however, she would be able to count down from 5 to answer the question  $5 - 3$  (removed items task) and say 4, 3, 2 ... also reaching the answer of 2. A child at stage 2 would not be able to do either the removed item task or the missing subtrahend task.



The last stage is Facile Number Sequence stage (stage 5) where a child can use a range of procedures flexibly to find the answer without counting by one. A child at this stage would be able to answer the problem:  $5 - ? = 3$  and  $5 - 3$  immediately, as recalled facts or using strategies based on inverse operations, 5 and 10 based number bonds, commutativity or compensation. A child at stage 5 is at the top of the SEAL 'ladder' concerning abstraction of number.

In my study I saw that the majority of learners were on stage 1 or 2 of the LFIN framework which means that they were on the count-all stages.

### **Pedagogic actions to promote abstract understanding**

In the remaining sections of this paper, I consider progression from concrete counting to abstract number objects in relation to specific resources and associated pedagogic actions. Within this discussion, I refer to resources that are increasingly available in the Gauteng classrooms that the broader Wits Maths Connect – Primary project is operating. My motivation for detailing these pedagogic actions with attention to the use of resources within these actions rests on evidence showing poor progressions within teachers' mediation of addition and subtraction tasks even when using structured resources (Venkat & Askew, 2012). This leads to their argument that:

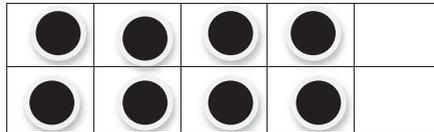
provision of artefacts that support more abstract ways of working with number is insufficient in improving teaching and learning. (p84)

Their argument follows from artefacts that were used with unit counting and therefore without reference to the 10-base structure of the abacus and the 100 square. There is also broader evidence of disconnections (Venkat & Adler, 2012) and ambiguity and highly 'localized' teacher talk in Foundation Phase classrooms (Venkat & Niadoo, 2012). My descriptions and discussions are therefore directed towards teacher development activities.

Overall, there is evidence that supporting teachers to use resources in ways that model progression, is needed. The table below shows a summary of key actions that teachers should be aware of when using these more structured resources, if they are aiming to support progression to more sophisticated strategies. I will discuss three structured resources and how they can be used to promote more sophisticated number strategies.

### Ten-frame cards

Ten-frame cards can be used to show ‘odd’ and ‘even’ number arrangements and to count through five. In the ‘pair-wise’ arrangement below, the aim is that children can ‘see’ that 2 is missing from the ten-frame card and that the number of dots is 2 less than ten which is 8, and not count the dots one-by-one up to 8.



### A 20-Bead-string (in multiples of 5)

20 beads (or bottle tops) with two different colours like below



The aim is to encourage children to show different numbers of beads with single movements. The bead string can also be used to ‘add through ten’. Example:  $7+5=?$ : use the bead string to ‘find 7’ by shifting 5 and 2 beads with one movement.



Then say that we know that it is 3 more to get to 10, which involves decomposing 5 into  $3+2$  and first add 3.



The last movement is the movement of two beads, because  $5=3+2$



Encourage children to ‘see’ that the answer on the bead string is  $10+2=12$ , without insisting that they must count out the answer one-by-one.



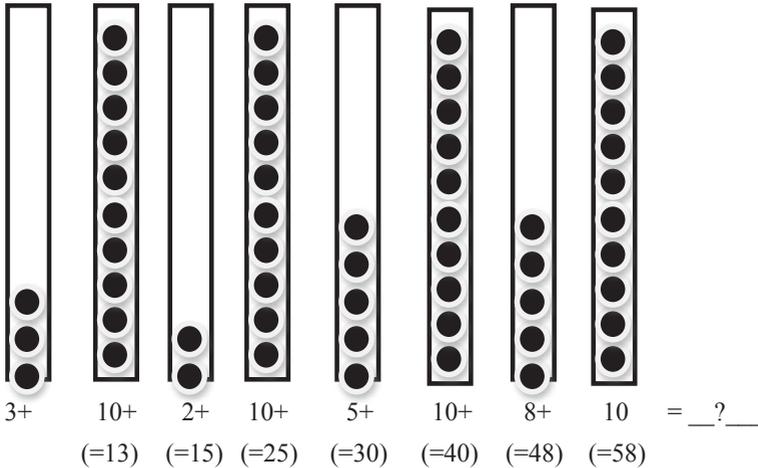
### Ten-strip cards

Ten-strip cards can be used to count-on and count-backwards in 10s, either ‘on’ or ‘off’ the multiples of ten. Teaching should establish first, by counting if needed, that one strip, as shown below, has ten dots – making it a ‘10-strip’.



Children must be encouraged to work with ten as an objectified quantity rather than needing to count in ones each time. Tasks with 10-strips involve laying out dot strips one at a time with the question: ‘How many dots?’ Encourage learners to count on from the previous total rather than reverting to the start each time.

Example:



The table below contrasts more concrete strategies with more abstract strategies in relation to each of these resources to highlight the differences.

<b>Resource</b>	<b>Concrete action/strategy</b>	<b>More abstract action/strategy</b>
10 frame cards	Counting the 8 dots on the cards using unit counting	'See' that there are two open spaces on the 10 frame card
Dot/domino cards	Counting the numbers on the cards using unit counting	Subitize the number of dots by seeing the number without counting them one by one requires familiarity with common dot and dice arrangements and grouped random arrangements
Bead string 1-20	Counting numbers in the 1-20 range using a unit count of beads	Demonstrating counting 'through 5' or 'through 10' with a single movement action
10 Strip cards	Counting the dots on the cards using unit counting	Adding the dots on a 10 strip card as a unit of 10 without counting them again

While above I have highlighted resources and pedagogic actions with them, in classrooms resources are not used in isolation of talk and learner response. Thus, to conclude this paper, I present a case of one learner's response to a set of early number tasks. I then detail the ways in which follow-up tasks with resources, can be used following diagnosis of the learner's SEAL stage, to help the learner to push forward. Thandeka is a pseudonym.



## Thandeka

Thandeka can say the Forward Number Word Sequence from 1 to 100. When asked to count on from 76 she skipped 77 and proceeded to 84. She counted on from 94-109 and then said 200, 201, 202, 203, 204, 205 and stopped when she was asked to stop. She can say the Number Word After (NWA) and Number Word Before (NWB) and for numbers 1-15 but not beyond that. Thandeka always counts from one when she adds two numbers. She cannot count backwards beyond 15. She can answer addition tasks where the sum is smaller than 10 but cannot answer  $9+6$ . When she was asked to add  $9+6$  screened counters, she counted out 9 on her fingers starting with 1 and tries to count-on and then gives answer of 16. When asked to add 8 and 5 screened counters, she gave the answer 18. When asked to find the answer to  $16-12$  she asked to use the counters. Counting out 16 counters, she then she took away 12 counters one-by-one. She then counted the remaining 4 from 1. She could subitize small single numbers. When the domino cards had 2 sides she had difficulties keeping track of the number of dots without counting them one-by-one, even when the numbers were smaller than 5.

### SEAL stage 2- Figurative Counting Stage

#### Follow-up pedagogic action:

1. To help her to count backwards beyond 15, Thandeka can be asked to repeat short BNWS initially using a number track after the teacher. Example: 'Say after me 18, 17, 16, 15, 14 pointing on the number track. Another way of helping her to count backwards is to say alternate number words backwards in the range 1-30, Example: teacher says 30, child says 29, teacher says 28, child says 27 ext.
2. To help her to subitize numbers of dots, the teacher must develop knowledge and more confidence with dice patterns 1-6 (subitising) by showing the domino cards repeatedly. To subitize the number of dots she has to 'see' the number of dots without counting them one-by-one. This requires familiarity with common dot and dice arrangements and grouped arrangements.
3. She is unable to use count on, to add numbers, so a first step would be to count-on in ones on the bead string and then moving to counting through ten.
4. To help her to add  $9+6$  the bead string can be used by counting through ten: example  $9+1=10$ . Decompose 6 to  $1+5$ , and then  $10+5=15$ . Another way to help her is to show combinations to 20 on a 20-bead-string

### CONCLUSION

Looking at the above, we know that children have to shift their thinking from a concrete understanding of numbers to an abstract way of working with numbers. Teachers must not always bring the children back to unit-counting if the child already made the shift and are ready to move to a next level of abstraction. Progression in number sense is possible and teachers need to focus intentionally to help children to think more abstractly about numbers.

## REFERENCES

New York: Cambridge University Press.

Anghileri, J. (2006). *Teaching Number Sense*. London: Antony Rowe.

Ensor, P., Hoadley, U., Jacklin, H., Kuhne, C., Schmitt, E., Lombard, A., et al. (2009). Specialising pedagogic text and time in Foundation Phase numeracy classrooms. *Journal of Education*, 23(1), 5-29.

Fleisch, B. (2008). *Primary Education in a crisis: why South African schoolchildren underachieve in Reading and Mathematics*. Cape Town: Juta and Company Ltd.

Haylock, D., & Cockburn, A. (2008). *Understanding Mathematics for Young Children*. London: Sage Publications.

Schollar, E. (2009). Towards Evidence-Based Educational Development in South Africa.

Sfard, A. (1991). On the dual Nature of Mathematical Conceptions: Reflections on Processes and Objects as Different Sides of the same Coin. *Educational Studies in Mathematics*, 22, 1-36.

Sfard, A. (1992). *Operational Origins of Mathematical Objects and the Quandary of Reification- The case of Functions*. America: Mathematics Association of America.

Sfard, A., & Linchevski. (1994). The gains and Pitfalls of Reification-The case of Algebra *Education Studies in Mathematics* (Vol. 26, pp. 192-228): Kluwer Academic.

Van der Berg, S., & Louw, M. (2006). *Lessons Learnt From SACMEQII: South African student performance in regional context.*: University of Stellenbosch, Department of Economics.

Venkat, H., & Adler, J. (2012). Coherence and connections in teachers' mathematical discourses in instruction. *Pythagoras*, 33(3), 8.

Venkat, H., & Askew, M. (2012). Mediating Early Number Learning: Specialising Across Teacher Talk and Tools. *Journal of Education*(56), 67-90.

Venkat, H., & Niadoo, D. (2012). Analyzing coherence for conceptual learning in a Grade 2 numeracy lesson. *Education as Change* 16(1), 21-33.

Weitz, M. S., & Venkat, H. (2013). Assessing Early Number Learning: How useful is The Annual National Assessment in Numeracy. *Perspectives in Education*, 31(3), 49-65.

Wright, R. J., Martland, J., & Stafford, A. K. (2006). *Early Numeracy* London: Sage Publications.



## THE ROLE OF TEACHERS IN DEVELOPING LEARNERS' MATHEMATICS DISCOURSE

Benadette Aineamani

Pearson Holdings South Africa

*This paper reports on work in progress on the role of teachers in developing learners' mathematics discourse. This paper therefore presents the rationale for the study, the theoretical framework, which will guide the study. The literature review and the research design and methodology, which will be used in the study, are also presented in this paper. In my previous study, it came to light that learners could not move back and forth between informal ways of communicating and formal ways of communicating without the teacher mediating the move. However, since the focus of my previous study was on the learners, I could not investigate the role of the teacher, hence my present study which will enable me to investigate how teachers play the role of developing learners' mathematics discourse*

### Introduction

Developing learners' mathematics discourse is a very important aspect in the teaching and learning of mathematics. According to a study conducted by Wachira, Pourdavood and Skitzki, (2013), an important aspect of any mathematics classroom in which learners are actively engaged, is to focus on discourse. The focus on mathematics discourse not only promotes the development of shared understandings and new insights but also contributes to a shift to more mathematical language in the classroom (Manouchehri, 2007). Chapman (2006) agrees with Wachira et al. (2013) that learning any mathematics concept requires a transformational shift from less mathematical language to more mathematical language. He purports that doing mathematics is the ability to say things in appropriate ways and appropriate forms. From Chapman's argument, if learners have not developed scientific mathematics discourse, then they are not yet able to do mathematics. For example, in order for learners to understand functions, they have to understand the formal language of functions.

According to Pimm (1991), learners bring informal mathematics language to the classroom, and this is the language learners tend to use to communicate their mathematics concepts and ideas. As discussed earlier, the same learners are expected to move from using informal mathematics language to a formal mathematics language, which is presented in the language of teaching and learning.

Therefore, it is very important that learners make the move from informal mathematics language to formal mathematics language since it is an expectation in the schooling system (Neria & Amit, 2004). Learners who make the move from informal mathematics language to formal mathematics language are said to have acquired control over the mathematics register as they are now able to talk and mean like mathematicians (Pimm, 1982). Halliday (1975: 65) defines a mathematics register as:

a set of meanings that belong to the language of mathematics (the mathematical use of natural language) and that a language must express if it is used for mathematical purposes. We should not think of a mathematical register as constituting solely terminology, or of the development of a register as simply a process of adding new words.

From the definition above, a mathematics register is more than just technical terms and vocabulary. It includes being able to argue and justify mathematics concepts using the language of mathematics. Therefore, it is important to develop learners mathematics discourse so that they are able to talk and mean like mathematicians.

In this study, I draw on Mortimer and Scott (2003) and Gee (2005) to define mathematics discourse. Gee (2005) defines discourse as language in use in any social setting such as a mathematics classroom while Mortimer and Scott (2003) define science discourse as talk which enables students to engage consciously in the dialogic process of meaning making and it is through science discourse that students acquire tools which they use to think through the scientific view for themselves. Therefore, from the definitions above, mathematics discourse in this study refers to the mathematics talk which learners and the teacher use to talk, agree or disagree about mathematics ideas and concepts in the dialogic process of meaning making in the mathematics classroom. A dialogic process is an ongoing process where learners compare and check their understanding of ideas with the ideas that are being rehearsed on a social plane. Meaning making is a dialogic process where learners bring together their ideas and work on them (Mortimer & Scott, 2003). In this study, I will focus on scientific mathematics discourse and everyday mathematics discourse.

Scientific mathematics discourse refers to the type of discourse where the learner uses scientific language to communicate mathematics concepts and ideas (Mortimer & Scott, 2003). This type of language is referred to as formal discourse, and this discourse uses specific talk or writing, using the LoLT (Setati, Alder, Reed & Bapoo, 2002). The terms used in scientific mathematics discourse are from the mathematics register.



For example, if a learner is asked to explain how to solve for  $x$  in this equation,  $2x^2 + 14x - 36 = 0$ , the learner has to refer to 2 and 14 as coefficients of  $x^2$  and  $x$  respectively, and -36 as a constant in the quadratic equation. This would be scientific mathematics discourse because the learner is employing words such as coefficient and constant, which are terms from the mathematics register. Everyday mathematics discourse refers to the type of discourse where the learner uses everyday language to communicate mathematics concepts and ideas (Mortimer & Scott, 2003). For example, if a learner is asked to explain how to solve for  $x$  in this equation,  $2x^2 + 14x - 36 = 0$ , the learner may refer to 2, 14 as numbers and not coefficients of  $x^2$  and  $x$  respectively, and refer to -36 as number and not a constant term in the quadratic equation. This is everyday mathematics discourse because the words used are not specific words from the mathematics register. The study will focus on the role of the teacher in helping learners move back and forth between scientific mathematics discourse and everyday mathematics discourse. According to Hedegaard and Chaiklin (2005), learning becomes very powerful if the teacher keeps in mind that everyday concepts and scientific concepts are very important and that they have to co-exist. In other words, teachers have to practice what they call a 'double move' (Hedegaard & Chaiklin, 2005), where everyday concepts are used to develop scientific concepts and scientific concepts are used to explain everyday concepts. Once learners understand and master the scientific language, they do not have to move back to everyday language (Hedegaard & Chaiklin, 2005).

The development of the mathematics discourse needs to be mediated because it requires the learner to master scientific concepts and these concepts are not acquired spontaneously or through the world of experience (Mortimer & Scott, 2003). The problem that makes my study worth investigating is that mediating scientific knowledge in a dialogic classroom is extremely hard (Brenner, 1994; Wachira et al., 2013). Developing mathematics discourse of learners in a mathematics dialogic classroom where learners' main language is not the language of teaching and learning (LoLT) is not an easy task (Brenner, 1994).

### **Purpose of the study**

The purpose of this study is to investigate the role of teachers in developing learners' mathematics discourse in a Grade 10 functions dialogic mathematics classroom.

## Research Questions

1. a) What kind of classroom environment do teachers set up in mathematics functions classroom?  
b) How does the environment set up support (or not) learner participation in the classroom discourse?
2. a) What discourses are used by teachers in the mathematics functions classroom?  
b) How does the discourse-in-use enable (or not) learners to move back and forth between informal and formal mathematics discourse in a functions class?

The research questions have been formulated using the language of the theoretical framework of the study.

## Theoretical Framework

This study situates itself within sociocultural theory and therefore draws on Mortimer and Scott's (2003) theoretical framework, which was developed to analyse meaning making in secondary science classrooms. There are socio cultural theoretical frameworks that were developed within mathematics classrooms. For example Nardi, Biza and Zachariades (2012) proposed a framework for analysing teacher and student arguments. They draw on Toulmin's model which describes the structure and semantic content of an informal argument. The limitation of Nardi et al's (2012) framework for my study is that it focuses on the analysis of teacher and student argument and not on the role of the teacher in developing learners' mathematics formal arguments, which I refer to as scientific mathematics discourse in my study. Therefore, both my study and Mortimer and Scott's framework focus on discourse and that is one of the reasons why Mortimer and Scott's (2003) framework is appropriate for my study. Mortimer and Scott's theory focuses on the role of the teacher in making the scientific theory available to learners and therefore I intend to use Mortimer and Scott's (2003) framework because it provides constructs such as teaching purpose and content of the lesson that explain how the teacher plays the role of making the scientific story available to learners. Mortimer and Scott (2003) also bring the teacher to the fore of their framework and argue that the teacher is of great importance in the science classroom. This framework is therefore appropriate for my study because I will be able to use it to interpret how teachers engage learners in mathematics talk and actions while acknowledging their everyday ways of talking, in an attempt to develop their scientific mathematics discourse. Part of the challenge in using Mortimer and Scott's framework is how it can be recontextualised in the mathematics classroom.

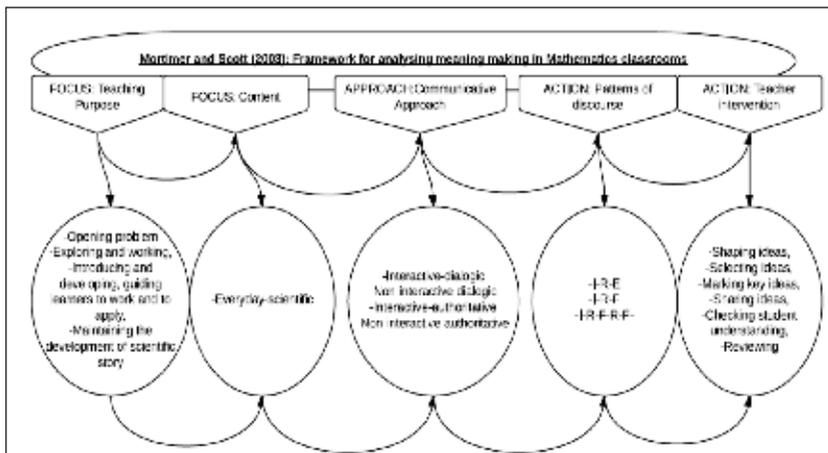


This is a challenge that I hope to engage with in the course of my study as I analyse the data obtained using the terms used in the framework. At the heart of Mortimer and Scott's (2003) theory lies Vygotsky's theory that all learning originates from social situations where ideas are shared through talk, gestures, writing, visual images and action and each individual is able to make sense of what is being communicated and the words used to communicate provide the tools for individual thinking (Mortimer & Scott, 2003). Mortimer and Scott's (2003) framework focuses on talking, meaning making and learning within the science discourse while my study focuses on mathematics discourse. Mortimer and Scott's (2003) framework is based on three aspects which are linked together. Mortimer and Scott's (2003) framework is based on three aspects which are linked together. The aspects are: focus, approach and action. Focus is divided into teaching purpose and content, approach aspect focuses on the communicative approach and then action is divided into patterns of discourse and teacher interventions (Mortimer & Scott, 2003).

<b>Focus</b>	1. Teaching Purpose	2. Content
<b>Approach</b>	3. Communicative approach	
<b>Action</b>	4. Patterns of discourse	5. Teacher interventions

Table 1: Aspect of analysis (Mortimer and Scott, 2003: 25)

The diagram below shows a summary of the theoretical framework of the study.



The theoretical framework will provide me with a lens to view the teaching focus of the lesson, the approach the teacher uses and the action the teacher takes in an attempt to develop learners' scientific mathematics discourse.

## Literature review

### Teaching and learning mathematics in a second or third language

Many learners find themselves being taught in a language which is not their first or second language (Setati, 2002; Adler, 2000), and this may need the teacher's careful intervention. In studies which were conducted in multilingual classrooms, teachers recognised that learners needed to learn the LoLT and the language of mathematics at the same time (Adler, 2000; Setati, 2002). For example in the study conducted by Adler (2000), instructions where the teacher was teaching both the LoLT and the mathematics language created what she called pedagogical dilemma for the teacher because the teacher did not know whether to teacher the LoLT or continue with the mathematics language. Adler argues that there is no solution to this dilemma but the teacher needs to find a way of managing the situation by being aware and using careful moves during instruction.

Learners in multilingual classrooms move between languages (Setati & Adler, 2000) and cultures (Zevernbergen, 2000). Cleghorn & Rollinick (2002) refer to the movement between the culture of the home and the culture of the school as 'border crossing'. Second language learners have to do a border crossing as well as moving between their home languages to LoLT and between informal and formal mathematics language. This, therefore, means they have to navigate between numerous social languages. In his study, Cuevas found that such learners may find it very difficult to follow the academic language which their teachers use to explain terms and concepts because the language does not relate to the informal talk which the learners bring to the classroom (Cuevas, 1984). The way learners make sense of the mathematics terms and concepts which the teacher is teaching is determined by their understanding of the language (Cuevas, 1984). Therefore, the way the learners will express or communicate their mathematics concepts will also be determined by the way they understand the language which the teacher uses to explain the terms and concepts. Barton and Barton (2005) conducted a study in four schools and one university over a two year period in New Zealand. One of the aims of the study was to examine the impact and nature of language factors in the teaching of mathematics learners for whom English is an additional language. They used observation, questionnaires and interviews for data collection. Twelve mathematics classrooms were observed and 16 students were interviewed.



They found that language features causing difficulties varied across the studies, and appear to depend on the mathematical level as well as the home language and English language proficiency levels. Vocabulary on its own was not the big issue that was anticipated in their study. However, it was a component of the difficulty experienced with understanding mathematical discourse as a whole. Mathematics that was integrated with everyday contexts also caused problems for learners that had difficulties with English as a language. Most learners in this study were unable to communicate their mathematics ideas in English but they were able to communicate the same mathematics ideas in their first languages, and some of the concepts lost mathematics meaning when communicated in their first languages.

### **The role of the teacher in developing learners' mathematics discourse**

Chapman (2006) argues that the teacher has to monitor the way learners use the language of mathematics within the classroom and the teacher has to emphasise, through classroom interaction, that learners use the appropriate language to communicate mathematics. For example, if a learner says 'the same', the teacher has to emphasise that the learner says 'constant'. For Chapman, the role of the teacher in a mathematics classroom is to emphasise the mathematics terms and ensure that learners use the appropriate language to talk about mathematics concepts and hence develop their mathematics discourse.

Wachira et al. (2013) argue for the need for teachers to show their learners that they value understanding the concepts rather than getting correct answers. Teachers need to manage the delicate balance of centralizing and guiding students' thinking while being careful not to overpower the meaning-making and dialogic process (Truxaw & DeFranco, 2007). In a study conducted by Staples and Colonis (2007) in the United States of America, they found that when the teacher allowed learners to respond to questions as a group, the learners stopped responding. When the same learners were given a chance to respond individually and explain their responses, learners became interested in responding to the teacher's questions and there was interaction between the learners and the teacher. Staples and Colonis (2007) also found that learners whose mathematics language skills were not developed struggled to take part in the interaction. Therefore, emphasising the need for classroom interaction may not be sufficient (Chapman, 2006). Deliberate teacher strategies may be required to help learners improve their mathematics language skills. Mercer and Sams (2006) conducted a study to explore the role of the teacher in guiding the development of learners' skills in using language in the mathematics classroom. The study involved 406 children and 14 teachers in schools in Milton Keynes in United Kingdom. Observations and formal assessment in experimental and control classes were used to collect data.

Data was collected through pre and post-intervention video recordings of a focal group in each target class. The study found that providing children with guidance and practice in how to use the appropriate language of mathematics would enable them to use language more effectively as a tool for working on mathematics problems together. However, in their study, Mercer and Sams (2006) did not discuss the different ways that learners can be guided by the teacher to develop appropriate use of mathematics language.

### Research design

In order to investigate the role of the teachers in developing learners' scientific mathematics discourse, I will conduct a qualitative study. According to Brantlinger, Jimenez, Klingner, Pugach and Richardson (2005: 195), qualitative research design is a "systematic approach to understanding qualities, or the essential nature, of a phenomenon within a particular research" Schumacher and McMillan (1993) argue that qualitative research views reality as multilayered and interactive. In other words, reality within a qualitative research is a shared experience which is interpreted by individuals who participate in the study. Opie (2004) also argues that main goal of a qualitative study is to understand the social phenomenon from the views of the participants. Therefore, a qualitative study seeks to understand people in their real settings and the reality of the settings changes with the changes in people's perceptions. Therefore, a qualitative approach is appropriate for my study because the study will take place in a real setting, the mathematics classroom, and the researcher will not try to manipulate the teachers' behaviour in any way.

The intention is to work with ten Grade 10 teachers in phase one of the study and then six Grade 10 teachers in phase two and phase three of the study. Phase one will be pre observational interviews, phase two will be classroom observations and then phase three will be post observation interviews. The teachers will be teaching the concept of functions to Grade 10 learners in one province in South Africa. I intend to conduct the study in Grade 10 classrooms because most of the mathematics terms in functions are introduced in Grade 10. Therefore, I intend to investigate how the teacher helps learners to develop their mathematics discourse while learning new terms and concepts. The teachers in the study will be selected from 10 schools.

In keeping with the sampling strategies of a qualitative study, purposive and opportunity sampling will be used in this study (Cohen, Manion & Morrison, 2000).



According to Merriam (1998), purposive sampling strategy is mainly used when the researcher is interested in discovering, understanding and gaining insight into a particular phenomenon. Purposive sampling is very important in my study because I will be able to select information rich cases, for example classrooms which are dialogic in nature. In other words, in line with my interest in investigating the role of the teachers in developing learners' scientific mathematics discourse, it is necessary that I observe lessons where the teachers are aware of their role in this regard. Thus, the main consideration will be in choosing teachers who express developing learners' scientific mathematics discourse as their goal during the lessons. Therefore, participants in my study will be selected purposefully, and the schools will also be selected purposefully. All the schools which will be selected in the study will have multilingual settings in the classroom in order for me to be able to give a rich description of how the teachers develop learners' scientific mathematics discourse in these settings. In summary, the sample for phase 1 of the study will be 10 teachers, the sample for phase 2 and phase 3 of the study will be 6 teachers as discussed earlier. I will take one week of regular visits to schools to familiarise myself with the schools before I start the data collection process. The period of data collection will take about six to ten weeks. The lessons will be about 50 minutes long and I will record lessons for each teacher from the start of the concept of functions topic to the end of the topic. Each teacher will have a total of 8 lessons. This will give me 48 video recordings of the whole study. As I record the lessons, I will take some field notes during the lessons in order to be able to capture as much information as possible.

## References

- Adler, J. (2000). *Teaching mathematics in multilingual classrooms*. Dordrecht, Kluwer Academic Publishers.
- Barton, P. & Barton, B. (2005). The Relationship Between English Language and Mathematics Learning for Non-native Speakers. *International Journal of Mathematical Education*, 36(7), 721-729.
- Brantlinger, E., Jimenez, R., Klingner, J., Pugach, M., & Richardson, V. (2005). Qualitative studies in special education. *Exceptional Children*, 71(2), 195-207.
- Brenner, M. E. (1994). A communication framework for mathematics: Exemplary instruction for culturally and linguistically different students. In B. McLeod (Ed.), *Language and learning: Educating linguistically diverse students* (pp. 233-267). Albany: SUNY Press.
- Chapman, O. (2006). Classroom Practices for Context of Mathematics Word Problems. *Educational Studies in Mathematics*, 62(2), 211-230.
- Cleghorn, A. & Rollinick, M. (2002). The role of English in Individual and Society Development: A View From African Classrooms. *Tesol Quarterly*, 36(3), 10-21.

- Cohen, L., Manion, L., & Morrison, K. (2000). *Research Methods in Education*. 5<sup>th</sup> edition. London and New York: Routledge Falmer.
- Cuevas, G. J. (1984). Mathematics Learning in English as a Second Language. *Journal for Research in Mathematics Education*, 15 (2), 134-144.
- Department of Basic education, (2010). *Curriculum and Assessment Policy Statement*, Government printer. Pretoria, South Africa.
- Gee, P. J. (2005). *An introduction to discourse analysis: theory and method*. New York: Routledge.
- Halliday, M.A.K. (1975). Some aspects of sociolinguistics. In E. Jacobsen (Ed.), *Interactions between linguistics and mathematical education* (UNESCO Report No. ED-74/CONF. 808, pp. 25-52). Paris: United Nations Educational, Scientific and Cultural Organization.
- Hedegaard, M. & Chaiklin, S. (2005). *Radical-local teaching and learning. A cultural-historical approach*. Arhus University press.
- Jamison, R. E. (2000). Learning the Language of Mathematics. *Language and Learning Across the Disciplines*, 4(2), 45-54.
- Manouchehri, A. (2007). Inquiry-discourse mathematics instruction. *Mathematics Teacher*, 101, 290-300.
- Mercer, N., & Sams, C. (2006). Teaching Children How to Use Language to Solve Maths Problems. *Language and Education*, 20 (6), 507-528.
- Merriam, S. B. (1998). *Qualitative Research and Case Study Applications in Education*. San Francisco, USA: Jossey-Bass Publishers.
- Mortimer, E. F. & Scott, P. H. (2003). *Meaning making in Secondary Science Classrooms*. Maidenhead: Open University Press.
- National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics*. Reston, VA: Author.
- Neria, D. & Amit, M. (2004). Students' preferences of non Algebraic representations in Mathematics Communication. In *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, Vol 3(27), pp 409-416.
- Opie, C. (2004). Research approaches. In C. Opie (Ed). *Doing Educational Research*. SAGE: London. Pp 73-94.
- Pimm, D. (1982). Mathematics? I speak it fluently. In A. Floyd (Ed.), *Developing mathematical thinking*, (pp. 139 – 150). Wokington: Addison Wesley.
- Pimm, D. (1991). Communicating mathematically. In Durkin, K & Shire, B (Ed.) *Language in Mathematical Education: Research and Practice*. Open University Press: Milton Keynes. Pp 17-23.
- Setati, M. & Adler, J. (2000). Between languages and discourses: Language practices in primary multilingual mathematics classrooms in South Africa, *Educational Studies in Mathematics* 43 (3), 243-269.
- Setati, M. (2002). Researching Mathematics Education and Language in Multilingual South Africa. *The mathematics Educator*, 12 (2), 6-20.
- Staples, M. & Colonis, M. M. (2007). Making the most of mathematical discussions. *Mathematics Teacher*, 101 (4), 257-261.
- Stathopoulou, C. & Kalabasis, F. (2006). Language and culture in mathematics education: Reflections on observing a Romany class in a Greek School. *Educational Studies in Mathematics*, 64(2), 231-238.



- Wachira, P., Pourdavood, R. G. & Skitzki, R. (2013). Mathematics Teacher's Role in Promoting Classroom Discourse. *International Journal for Mathematics Teaching & Learning*, preceding p1-38.
- Zevenbergen, R. (2000). "Cracking the Code" of mathematics classrooms: School success as a function of linguistic, social, and cultural background. In J. Boaler (Ed.), *Multiple perspective on mathematics teaching and learning* (1st Ed.), pp. 201 – 242. London.

## ENGAGING WITH COGNITIVE LEVELS: A PRACTICAL APPROACH TOWARDS ASSESSING THE COGNITIVE SPECTRUM IN MATHEMATICS

Ryna du Plooy, Caroline Long

University of Pretoria

*In the interest of further progression beyond the Intermediate Phase, conceptual mastery of various crucial mathematical ideas is necessary. If teaching, learning and assessment remain on the factual recall and operational efficiency levels, progression to the more advanced levels of mathematical activity is severely restricted. In this paper we advocate the integrated assessment of conceptual understanding, across the dimensions of mathematical understanding and on different cognitive levels. We propose a framework for teachers to use in constructing classroom assessments that accommodate the dimensions of understanding at various cognitive levels, to prepare learners for the Senior Phase.*

### FOCUS AND SIGNIFICANCE OF THE PAPER

This paper is the result of du Plooy's research quest for a practical and reliable way to comply with the CAPS requirement to assess mathematics at different cognitive levels. The study is supervised by Long. In this study some principles informing mathematics teaching and learning design which use specific metacognitive strategies in division at the Intermediate Phase (Grade 6) are investigated.

What emerged while developing the research instruments for the research design was the necessity to enter into the intervention phase with a clear picture of the participants' competency. Firstly, a baseline assessment had to be set enabling a diagnosis of the existing voids in the participants' understanding of the multiplicative structures in real life situations. Secondly, a pre-intervention assessment and a post-intervention assessment had to be set, if the effect of using the proposed metacognitive strategies was to be investigated.

The insights in this paper originated from informal action research in classroom practice and were theoretically grounded through a formal literature review. The practical application of this research finding, culminating in a framework, is that teachers can use this framework to set and evaluate their own assessments according to the requirements of the Intermediate Phase curriculum, for baseline, diagnostic, formative and summative assessment, but particularly to understand how assessment of cognitive levels is done practically within a real classroom setting.

### THEORETICAL ASPECTS OF THE FRAMEWORK

A number of theoretical and practical aspects were taken into account while designing the assessments for the study:

- The South African mathematics curriculum. The Curriculum and Assessment Policy Statement (CAPS) (Department of Basic Education (DBE), 2012) was the point of departure in the study;
- The CAPS (DBE, 2012) spells out affective and cognitive goals for mathematics learning. This research is situated in the cognitive domain of mathematics teaching and learning and therefore had to take into account the curriculum requirement of ascribing cognitive levels in assessment (DBE , 2012, p. 296);
- The theoretical antecedents for the cognitive levels used in CAPS were studied in order to apply them in setting the assessments. Similarities were found between the approach in CAPS and the revised form of Bloom's taxonomy for learning, teaching and assessing (Anderson & Krathwohl, 2001; Anderson, 2002). In the revised taxonomy, Anderson and Krathwohl (2001) describe Bloom et al.'s original categories (factual knowledge, conceptual knowledge and procedural knowledge) in action words, and add the metacognitive category of knowledge to the original three categories;
- Mathematics is a complex subject which requires an intricate process of teaching and learning, and multidimensional understandings (Usiskin, 2012) such as understanding the various mathematical representations, the properties of mathematical concepts, the application of operations within certain problems and the understanding of algorithms or methods. Assessment would therefore need to be articulated according to these dimensions of understanding, in addition to the levels of understanding proposed by Anderson and Krathwohl (2001).

## THE RELATED LITERATURE

For the purpose of this paper, we focus primarily on the literature regarding cognitive levels of mathematics understanding with a secondary focus on a single selected theory of mathematics understanding, as follows:

### Cognitive levels of mathematics understanding

Apart from the different mathematics areas, the CAPS (DBE, 2012, p. 296) also describe four cognitive levels at which assessment has to be conducted. These levels are: knowledge (25%), routine procedures (45%), complex procedures (20%) and problem solving (10%). The four cognitive levels used in CAPS, correspond directly with the Subject Assessment Guidelines of the 1999 TIMSS taxonomy of categories of mathematical demand (Stols, 2013, p. 13). The cognitive categories used in this international test need to be interpreted carefully if they are to be applied in classroom assessment.

For the mathematics teachers, as the users of CAPS, these categories leave room for individual interpretation when an assessment is set. We therefore identified a need for clarity about the concept of cognitive levels as it is explained in the literature.

Linn (2002, pp. 28-37) uses the term “level of cognitive demand” for what CAPS terms “cognitive levels”. He discusses the approach taken by the International Assessment of Educational Progress (IAEP), using the three cognitive levels conceptual understanding, procedural knowledge and problem solving. He points out the similarities between the IAEP approach and the five cognitive process categories called performance expectations in TIMSS 1995, namely understanding, routine procedures, and problem solving, investigating and communicating.

It was however Anderson and Krathwohl’s (2001) revision of Bloom’s taxonomy of educational objectives, that helped the first author to conceptually delineate cognitive levels in assessment. Of the four cognitive categories, namely factual knowledge, procedural knowledge, conceptual knowledge and metacognitive knowledge, the first three have been used in the [assessment matrix](#) in Table 1 (Dimensions of understanding and levels of understanding, du Plooy, 2014). The term “factual recall” is used for the first level, meaning “to bring to the fore, knowledge aspects that have been stored in the learners’ memories”. For the second level “procedural efficiency” is used, adopting Hiebert and Carpenter’s (1992) description of procedural knowledge as “a sequence of actions...the manipulation of written symbols in a step-by-step sequence” (p. 78). The third level is termed “*conceptual grasp*”. This use of the term is in direct contrast with Feuerstein’s (Feuerstein & Rand, 1974, Feuerstein et al, 2006) construct of “*episodic grasp of reality*”, where the latter refers to the incidental and isolated experience of reality, unrelated in time and space to other experiences. The construct “conceptual grasp” as used in this study denotes the interrelatedness of an idea with other ideas to form a coherent mental unit, which can be conceptualized as one concept, but large enough to contain different sub-concepts. “Rate” as a mathematical concept is a good example, with speed, unit price, population density and so on, as sub-concepts (see par 3.2).

### **Usiskin’s theory of mathematical understanding**

Mathematical activity, according to Usiskin (2012, pp. 2-3, 19) consists of concepts and problems. Within a mathematical problem, several concepts would come to the fore, as is demonstrated in this paper. Usiskin explains that a concept can be used as the unit of analysis, and it is of such a nature that it can be analysed systematically (Usiskin, 2012, p. 15). Usiskin’s perspective applies to the research of a concept, and when I interpret it from a teaching point of view, a concept may then be seen as a mathematical unit that can be taught and learned systematically.



Furthermore, Usiskin distinguishes five dimensions<sup>1</sup> (Usiskin. 2012) that constitute mathematics understanding, of which I use four in my assessment matrix of understanding in Table 1 du Plooy, 2014), namely the skill-algorithm understanding, the property-proof understanding, the use-application understanding and the representation-metaphor understanding. In this paper I use this distinction as described by Usiskin (2012, pp. 4-9), as follows:

- The skill-algorithm dimension or procedural understanding essentially entails knowing *how* to get an answer.
- The property-proof dimension involves the identification of the mathematical properties that underlie *why* the particular method of obtaining the answer worked.
- The use-application dimension implies knowing *when* to apply a specific operation.
- The representation-metaphor dimension requires the ability to communicate a mathematical concept by means of an appropriate representation.

### **The importance of mathematics learning on all cognitive levels**

Pantziara and Philippou (2011) draw upon existing studies to make a broad distinction between procedural and conceptual knowledge of mathematics (pp. 61-83). Their research reveals that learners have a better command of some of the problematic mathematical areas like fractions if conceptual knowledge has been developed alongside procedural knowledge. Voutsina (2011, p. 196) describes the relationship between procedural- and conceptual knowledge, and problem solving skills as an iterative process. She found that changes in young children's problem solving behaviour stem from the dynamic interaction between procedural and conceptual knowledge.

Inferring from the literature and du Plooy's own experience it seems entirely plausible that failure to make the cognitive transition from knowledge and routine procedures to complex procedures and problem solving, may account for deteriorating mathematic competence at the more senior levels.

Although factual knowledge and procedural efficiency are important constituents of mathematics proficiency, the transition to the more advanced cognitive levels does not automatically follow on the direct recall of mathematical facts or even on computational efficiency (arriving at the correct answer) in mathematical procedures. This transition rather hinges on true and multidimensional understanding of mathematics concepts. The teaching, learning and assessment processes relating to the conceptual grasp of mathematical constructs are therefore crucial for mathematics progression.

---

<sup>1</sup> The fifth dimension, the cultural-historical dimension, is omitted, as it is not relevant in the context of this study.

Subsequently, mathematics assessments have to be created in such a way that they provide for the demonstration of conceptual grasp in addition to factual recall and operational efficiency.

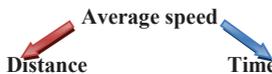
### AN ILLUSTRATION OF THE METHOD USED IN THIS PAPER

In setting the assessments it was useful to evaluate test items against the four dimensions of understanding, thus ensuring representivity across the spectrum of mathematics understanding. Furthermore, in the practice of doing this, the need was soon experienced to express, within a specific dimension of understanding, the level of understanding needed to succeed in solving a mathematical problem.

A matrix in Table1(du Plooy, 2014). was subsequently developed according to which test items could be plotted, not only in terms of its dimension of mathematical understanding, but also on the cognitive level that the problem requires. In this matrix Usiskin’s (2012) four selected dimensions of understanding are juxtaposed with the first three cognitive levels of Anderson and Krathwohl(2001) as explained above, resembling the *IAEP Mathematics Framework for 9- and 13-Year-Olds* (Linn, 2002, p.35). Anderson and Krathwohl’s metacognitive level is not used, since it is not used in the CAPS. The use of this matrix is demonstrated on a test item based on a real-life problem as it would be posed to a Grade 6 learner, as follows:

“Ms Buti drove from Bela-Bela to Bethulie. She started from Bela-Bela at 9:30 am, with the odometer of her car on 88 888 km. She arrived in Bethulie at 16:30, with the odometer on 89 525 km. What was the average speed of Ms Buti’s journey?”

The main concepts involved in the above problem can be set out as follows:



“Analogue time” and “digital time” are additional concepts without which the problem cannot be solved, because the problem statement makes use of both.

In the example above, we question whether the factual recall of the term “average speed” on its own can ensure the successful solution of the problem. Here much more understanding is required. We now analyse the above mathematics problem according to the dimensions of understanding (Usiskin, 2012), relating to each concept to the level of understanding that it requires. Broad guidelines are given below as an indication of some possible plotting of concepts onto the matrix, but through an interactive session we shall enrich this list and amend it with more content on each level:

The concept “average speed” requires multidimensional mathematical understanding on different cognitive levels:



- a. To understand the meaning of “average speed” and the elements it entails

Cognitive level:	factual recall
Mathematical dimension:	representation-metaphor
- b. To understand the elements that “average speed” entails

Cognitive level:	factual recall
Mathematical dimension:	property-proof
- c. To understand which operation to use to calculate the average speed
 

Cognitive level:	factual recall, conceptual grasp
Mathematical dimension:	use-application
- d. To understand how to calculate “average speed”

Cognitive level:	operational efficiency
Mathematical dimension:	skill-algorithm

The concept “distance” needs multidimensional understanding:

- a. To understand the role of “distance” in calculating average speed
 

Cognitive level:	factual recall
Mathematical dimension:	representation-metaphor
- b. To understand which operation to use in calculating the distance
 

Cognitive level:	conceptual grasp
Mathematical dimension:	use-application
- c. To understand how to calculate “distance” correctly
 

Cognitive level:	operational efficiency
Mathematical dimension:	skill-algorithm

The concept “time” needs multidimensional understanding:

- a. To understand the role of “time” in calculating average speed
 

Cognitive level:	factual recall
Mathematical dimension:	representation-metaphor
- b. To understand which operation to use in calculating the time using the available information
 

Cognitive level:	conceptual grasp
Mathematical dimension:	use-application
- c. To understand how to calculate “time” correctly
 

Cognitive level:	operational efficiency
Mathematical dimension:	skill-algorithm

Following such an analysis, we can, by way of a practical exercise with participants, map the above mathematics problem on a matrix according to the dimensions of understanding relating to each concept (Usiskin, 2012), at the level of understanding that it requires, as follows:

<b>Dimensions of understanding</b> (Usiskin, 2012)				
<b>Levels of understanding</b> (Adapted from Anderson & Krathwohl, 2001)	<b>Use / application</b>	<b>Skill / algorithm</b>	<b>Representation / metaphor</b>	<b>Property / proof</b>
	<u>Factual recall</u> (the rule applicable to the calculation of average speed)	<u>Factual recall</u> (eg steps in the division calculation)	<u>Factual recall</u> (the meaning of average speed)	<u>Factual recall</u> (attributes of average speed)
	<u>Operational efficiency</u>	<u>Operational efficiency</u> (calculating towards a correct answer)	<u>Operational efficiency</u> (formulating an answer km/h)	<u>Operational efficiency</u>
	<u>Conceptual grasp</u> (this situation requires division)	<u>Conceptual grasp</u>	<u>Conceptual grasp</u> (e.g. what rate entails)	<u>Conceptual grasp</u>

Table 1: Dimensions of understanding and levels of understanding (du Plooy, 2014 ©)

The above matrix is proposed as a tool in ensuring the representivity of an assessment item across the spectrum of mathematics understanding, on all cognitive levels required for thorough understanding of a concept. It has the potential of extending the present context to a more inclusive concept like “rate”. In fact, the transition to the Senior Phase and its increased emphasis on more abstract concepts, make the conceptual grasp of the idea of “rate” an absolute necessity.

## REFERENCES

- Anderson, L. W., & Krathwohl, D. R. (Eds.). (2001). A taxonomy for learning, teaching and assessing: A revision of Bloom’s taxonomy of educational objectives: Complete edition. New York: Longman.
- Anderson, L. W. (2002). Curricular alignment: A re-examination. *Theory into Practice*, 41(4), 255-260. Retrieved from [http://www.unco.edu/cetl/sir/stating\\_outcome/documents/Krathwohl.pdf](http://www.unco.edu/cetl/sir/stating_outcome/documents/Krathwohl.pdf)
- Department: Basic Education South Africa. (2012). National curriculum statement (NCS). Curriculum and policy statement: Intermediate Phase. Grades 4-6. Mathematics. Pretoria, South Africa: Author.
- Feuerstein, R. & Rand, Y. (1974). Mediated learning experiences: An outline of the proximal etiology for differential development of cognitive functions. In *Journal of the International Council of Psychology*. 9/10. 7-37.
- Feuerstein, R., Falik, L. H., & Feuerstein, R. Jnr. (2006). Definitions of essential concepts & terms, a working glossary. Jerusalem: The International Center for the Enhancement of Learning Potential.



- Hiebert, J. & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning: a project of the National Council of teachers of mathematics*. 65-97. New York: Macmillan.
- Linn, R. L. (2002). The measurement of student achievement in international studies. In A. C. Porter & A. Gamoran (Eds.), *Methodological advances in cross-national surveys of educational achievement*. Washington: National Academy Press.
- Pantziara, M. & Philippou, G. (2011). Levels of students' "conception" of fractions. *Educational Studies of Mathematics*, 79. 61-83. Springer Science and Business Media.
- Schoenfeld, A. H. (1994). *Mathematical thinking and problem solving*. London: Routledge.
- Stols, G. (2013). An investigation into the opportunity to learn that is available to Grade 12 mathematics learners. *South African Journal of Education*, 33(1), 1-18. Retrieved from [www.ajol.info/index.php/saje/article/viewFile/85005/74976](http://www.ajol.info/index.php/saje/article/viewFile/85005/74976)
- Usiskin, Z. P. (2012). What does it mean to understand some mathematics? 12<sup>th</sup> International Congress on Mathematical Education held in Seoul, Korea, 8-15 July 2012, 502-521. Seoul, Korea: COEX.
- Vye, N. J., Delclos, V. R., Burns, M. S., & Bransford, J. D. (1988). Teaching thinking and problem solving: illustrations and issues. In R. J. Sternberg & E. E. Smith, *The psychology of human thought*. 337-366. Cambridge: University Press.
- Voutsina, C. (2011). Procedural and conceptual changes in young children's problem solving. *Educational Studies of Mathematics*, Vol. 79, 193-214. Springer Science and Business Media.

## TEACHER AGENCY AND PROFESSIONAL PRACTICE: DEVELOPING AND NURTURING CREATIVITY IN MATHEMATICS TEACHER EDUCATION

Caroline Long

University of Pretoria

Erna Lampen

University of the Witwatersrand

*In this paper we reflect on the current problems in teacher education as identified in recent research and policy documents, notably the Integrated Strategic Planning Framework for Teacher Education and Development in South Africa. We then offer a critique of the problem and of the solutions proposed in policy documents, by drawing on educational theory. In particular we draw on educational philosophy regarding the vocation of teaching and a theory of professional agency, to propose an alternative set of problems with proposed solutions. We deliberate on what this alternative means for mathematics education in terms of teacher education development. We explore the complementary roles of the university education departments, the national Department of Basic Education, professional mathematics education organisations, and the ancillary role of business.*

### INTRODUCTION

The indicators of “effective” education in general, and mathematics education in particular, often revolve around technical issues that can be monitored fairly easily, for example the number of pages learners have completed in an exercise book. Another issue that appears to be easily monitored by regular systemic testing is learner proficiency in mathematics and, it seems, teacher knowledge of mathematics. For example, the first eight pages of the Mpumalanga Education Department Mathematics Strategy (Mpumalanga Department of Education, 2014) consist of systemic test results, with little explanation of what exactly was being tested. Teacher knowledge is evaluated in terms of answers to test items designed for systemic purposes, and mostly items divorced from the context of the classroom. In the same *Strategy* document learner proficiency is associated with a score on the Annual National Assessments. These tests, comprising 30 to 50 items intended to cover the entire mathematics curriculum and with each item requiring an answer in two and a half minutes, require a quickness which is not necessarily associated with thoughtful action.

Less easily measured indicators are those relating to the deeper issues of the purpose of education. What measures should be put in place to gauge the influence and contribution a mathematics teacher has had on the passion and excitement of a third year university student who spends 16 hours a day pursuing his passion, quantum mechanics? What elements of his school education have propelled this student from a desk in a poorly resourced Mpumalanga classroom, to the physics laboratories of a top South African university?

(Wylde, 2011?). Might we surmise that this mathematics teacher had vision, had agency, and saw his vocation as making a difference in the lives of his students? Such a teacher could be described as a rich and invaluable resource to his/her learners. In this paper we argue that we need mathematics teachers who are rich resources, adults with vision and agency who can inspire learners through their teaching to reach beyond current confines and prepare them to succeed in future learning.

The essential argument follows four steps. Firstly, we all agree that effective teaching of meaningful content is a prerequisite for better outcomes. Secondly, the teacher is critically important in the educational process. Even in the age of technological advances and easy access to information, teachers provide role models for learners and become mentors as they guide and accompany learners toward a successful exit from the school system. And then the fourth point is that teacher agency is hypothesized here to be closely associated with effective teaching.

With the above argument informing our lens, we investigate the problems and solutions proposed in the Integrated Strategic Planning Framework for Teacher Education and Development in South Africa (DBE, 2011).

### **The problem according to the Framework**

The problem according to the Framework comprises three main aspects: firstly the “*failure of the system* to achieve dramatic improvement in the quality of teaching and learning”, secondly, a “*fragmented and uncoordinated* approach” to teacher development, and thirdly, the *tenuous involvement of teachers*, teacher organisations and stakeholders in improving the system (DBE, p.1, authors’ emphasis).

The first problem immediately invites a further question, “What is the evidence for the failure of the system to achieve dramatic improvement in the quality of teaching and learning?” The evidence proposed in the document for this “failure” of the system is the results of systemic and international tests. “System” we think refers to the complete education organization. A straight line to cause, is drawn from the system results, based on perceived effect, back to events in the classrooms, to the preparedness of teachers, and then to the efforts of the department of education to enhance preparedness and finally back to the content and aims of the mathematics curriculum. Here we contest the validity of test scores as an evaluation of the education system. We also note here that the world has changed “dramatically” over the past 30 years. This change from an industrial world to an information-driven world has brought about a different set of criteria for proficiency in the wider world, and across the world education systems are working hard at developing answers to new educational demands. We acknowledge here that elements of systemic assessment, that is the numeracy and literacy items testing some basic knowledge, constitute part of what is needed, however the decontextualized nature of the assessment and the items, may not provide valid indicators of mathematical proficiency of either learners or teachers (Schoenfeld, 2007).

The second problem is identified as a “fragmented and uncoordinated” approach to teacher education development. What is the evidence for this statement? On the one hand it seems that the reasoning behind the formulation of the problem is that the roles and the responsibilities of teachers outlined in the Norms and Standards for teachers (DBE, 1998) do not serve as magnets for consolidation and coordination for teacher education. If, on the other hand it suggests that the variation between the teacher education programmes at the different universities are problematic, then we disagree. The first point of disagreement is that the purpose of university teaching is not to produce drone teachers that can be used to deliver messages from external agents. The purpose is rather to transform students to become independent thinkers and actors (Biesta, 2009) who can adapt to a changing world and teach learners to adapt. This purpose requires that teachers themselves be independent and creative thinkers. We acknowledge here that the development of teachers’ mathematical proficiency (Kilpatrick et al., 2000) is definitely the current priority of the teacher education departments in South Africa (and elsewhere); the question rests on how this proficiency is defined and attained.

The second point of disagreement, in alignment with the work of Griffiths (2013), is that however “good” the teacher education department (of the university), it is in the first five years of teaching that the teacher develops into the vocation of teaching. It is not so much the teacher’s performance in front of the class, demonstrating her own knowledge of mathematics, but her engagement with the learners in order to develop their mathematical knowledge and reasoning that is a measure of “teacher effectiveness”.<sup>1</sup> As teacher educators we do not absolve ourselves of the responsibility to reflect on our current practices. On the contrary, we question the extremely narrow focus on knowing the mathematics in the curriculum. This focus tends to exclude knowing mathematics in such a way that enables a teacher educator to involve prospective teachers with reasoning about the mathematics in the curriculum beyond the confines of current sequencing of topics. We see our responsibility as engaging with the national Department of Education concerning effective undergraduate teaching of mathematics for prospective teachers.

The third problem identified is the “tenuous involvement” of teachers, teacher organisations and stakeholders in improving the system. The question arises here concerning the categories included under “teacher organisations” and “stakeholders”. Do the teacher organisations include professional organisations such as AMESA, in addition to the obvious candidates - the teacher unions? And who are the stakeholders? Since there is no direct reference to teacher educators, we assume they are stakeholders, but their status and agency as stakeholders are not clear.

---

<sup>1</sup> The term effectiveness has been critiqued by Biesta (2009a) as it is meaningless without a clear sense of purpose. For example if the performance on the system results improve (a measure of effectiveness), but the learners are sitting mesmerised in their desks hanging on to the teacher’s every work with no critical engagement, then we question the effectiveness of the education.



We would expect that the teacher educators, like teachers, be acknowledged as critical in the initiating of teachers into the vocation of teaching. By contrast, the business model<sup>2</sup>, for example the National Education Collaborative Trust (NECT) which advocates “fixing” the education system “once and for all”, pays little attention to the critical role of teachers.

Where teachers are explicitly mentioned and seen to play a role in improving “the system”, we reflect on the way teachers are portrayed in the document. The current Framework places the teachers “firmly at the centre”, and responsible for their “own development” (DBE, 2011, p. 1). This statement is somewhat contradictory, as on the one hand the teachers are being trusted to attend to their own needs and that of their learners as they see fit, and as befits the context in which they find themselves, but on the other hand teachers and their students are tested on a centralized curriculum and a one-size-fits-all test into which they had limited input. Again we reiterate that a test score does not constitute an evaluation of a complex practice like teaching.

### **The solution according to the Framework**

According to the Framework, the first outcome is to “(i)mprove the quality of teacher education and development in order to improve the quality of teachers and teaching” (DBE, 2011, p. 4). Output 1 of this outcome is to identify and address “individual and systemic developmental needs”. Three goals are identified here, the first is to “establish an institute”, the second is to “develop and deliver teacher diagnostic self-assessment”, and the third is to “develop high quality teacher resources” (DBE, 2011, p. 4).

The proposed National Institute for Curriculum and Policy Development (NICPD) is responsible for “developing and managing a system for teachers to identify their developmental needs and access quality developmental opportunities ...” (DBE, 2011, p. 5). Are we to have yet another body of highly paid officials whose purpose is to manage teachers? As regards the “teacher diagnostic self-assessment” we wonder whether a “test” of 30 questions with each question to be answered in less than 120 seconds is helpful at all to a teacher who may know the mathematics to be taught, but who struggles to engage learners in meaningful discussions for the purpose of developing mastery over the discourse (Wagner, 2007).

The value of the third goal, developing high quality resources, depends on the engagement of the teachers with the developing of external resources suitable to their context. Will these resources take the form of lesson plans, complete with worksheets for the learners? Many excellent resources are already available from the internet. Yet, the selection of the most suitable resources is the challenge for teachers, and indeed for the education department.

---

<sup>2</sup> “Big business and the government have committed almost R600m to a partnership initiative which could provide a model to fix South Africa’s “fragile” education system “once and for all”, says Basic Education Minister Angie Motshekga”. Marrian and Joffe, Business Day, February 06 2014

The key question is what kind of quality resource places the teacher at the centre of system, and what kind strips her of agency and hence defers her responsibility for her learners' development.

While teachers are “at the centre of the system” and must take responsibility for their own learning, there is currently very little room for exercising professional agency in the mathematics classroom. Firstly, the time allocations in Curriculum and Assessment Policy Statement (CAPS) are very specifically and tightly dictated. The sequencing of concepts is firmly fixed with the expectation that in the classroom a particular order is followed. There are reports that the Grade 6 CAPS curriculum is too full. There are too many new (fragmented) concepts in one year. In personal communication a mathematics teacher with 20 years' experience noted that the separation between and sequencing of fractions, decimals and measurement in the intermediate phase was problematic. The difficulty then is for her as a teacher in a controlled system to exercise her professional judgment, or professional agency, and sequence this set of related concepts in a way that makes sense in her classroom, but also serves the interest of the system, such as readiness for systemic assessment.

## THE TEACHER AND TEACHING

We take the philosophical standpoint that teaching is a vocation; it is unusually worthwhile and important. Teachers in general have a “natural urge to take care of others” and “contribute to the betterment of society” (Higgins, 2013). Teachers want to “make a difference”. Many, if not most, teachers want to make a difference in the lives of their learners. They are aware of the lifelong consequences of their teaching for their learners, either good or ill. South African teachers are no different when they speak out about their aims and dreams. If they are judged to fall short as teachers, we should investigate what robs them of their professional agency.

### Four intersecting roles of the teacher

According to Griffiths (2013) the teacher has to coordinate at least four roles inside her classroom. The first is her role in the *dyadic relationship* between the teacher and the individual student. In this role she has to be constantly aware of individual needs for differentiation in tasks, in discussion, in support for some and to extend others. The second is her role in the (cognitive and physical) *organization of the class*, to establish an association with a common goal. In this role a teacher has to decide in the moment whether individual needs can be subsumed in the larger project of taking a class forward, or whether the class as a whole must hone in on the problem of an individual learner. The third is an *instrumental role*, to ensure that goals are met. Here we note that the practical day to day requirements of doing homework and ensuring the learners submit their assignments on time, together with the demand to keep to the timeframes of tests and examinations, are all part of what it means to be a teacher. The fourth role is coordinating the *triadic relationship between the subject matter, the teacher and the learner*.



The excitement and passion for the subject that is shared by both the learner and the teacher is an essential element of teaching – instilling a love for the subject. The extending of learners interests beyond the classroom, for example the participation in mathematics Olympiads, or organizing trips to astronomy observatories, may just spark a passion for some learners.

We propose that keeping the above four roles in balance in a class of 30 learners (or more) is a challenging task. Yet, the expectation is that undergraduate teacher education has to prepare a mathematics teacher to take up these roles from the day they enter their first classroom. The construct “teacher agency” is, we believe, the keystone to preparing teachers to deal with these many demands, and other demands which come from outside the classroom. Teacher agency, like a keystone holding in place the lateral stones to form a structurally sound arch, holds in place the myriad demands and roles.

### The concept of agency

The concept of agency is here defined as the dynamic competence of human beings to act independently, and to make choices (Priestley, Biesta & Robinson, 2013) in order to advance toward their goals. Two additional ideas are key to this concept: the first is that agency is not intrinsic to a person, but rather perceived as occurring interactively with the environment, and secondly that the environment in which individuals find themselves may enable or constrain agentive action (Biesta & Tedder, 2006).

Colapietro (2009) distinguishes between two models of human action, namely the *model of rational action* and that of *normatively oriented action*. Normatively oriented action<sup>3</sup> consists of ideals, standards and values provided by society which are used to evaluate one’s own and behavior and that of others (Gone, Miller, & Rappaport, (1997). When interpreted from the perspective of a teacher this action would involve following the codes set down by the department and the school. Rational action is understood as the consideration of beliefs and goals in order to determine a course of action (Goldthorpe, 1998). Here we envisage the distinction between the demands of authority and the reasoned actions judged to be in the best interests of the learner.

According to Colapietro (2009, drawing on Joas, 1996) a third category is needed, that of creative activity. This category supersedes the former categories and describes the human agency in the world. If we surmise that effective teaching and teacher agency are highly correlated, the question arises as to which components of a curriculum for initial mathematics teacher education will enable agency and therefore effective mathematics teaching.

---

<sup>3</sup> “ Normative Orientation consists of those criteria (ideals, standards, values) provided by culture which are used in self-evaluation and the judgment of one’s own conduct and the conduct of others (Gone, T., Miller, P. & Rappaport, J. (1997).

## TEACHER EDUCATION: DESIGN AND RESPONSIBILITY

Both the design of teacher education courses and the responsibility for teacher education are currently contested terrains. From the Framework the role of university education departments is not clear. “Stakeholders” we think include the universities, business and the government, though this is not clear. Increasingly big business is being called on to “fix education” “once and for all”. In this section we discuss the design of teacher education courses, bearing in mind the keystone that we surmise holds complex demands in place, and the organisations responsible for delivering education, the universities, national departments, professional organisations and business.

We concur with Batra (2009) that teacher agency or empowerment is necessary to achieve both quality education and the associated social transformation. Batra admonishes the Indian education system for not taking seriously the professional development of the teacher since attaining independence from colonial rule in 1948. She cites the invisibility of the teacher in curriculum design and in the formulation of education policy. In some respects the two countries have undergone parallel experiences, though we acknowledge that the input of professional organisations in South Africa has played an important role in policy development and some curriculum implementations.

Batra (2009) contrasts two perspectives on the teachers role in the broader social context, the first is that of an *implementing agent* of the current economic needs and the social needs of the country. This role is somewhat aligned with the “drone model” or the NECT model whose aim is to “fix education once and for all”. An alternative perspective is that of an agent of change. In this role the teacher becomes the pivot for social transformation. Her role involves engaging directly with children, and how they are thinking. This role aligns with Griffith’s dyadic function where the teacher engages with the individual student. A second important aspect for the agent of change conceptualization is that the teacher understands the relationship between school and society and in this way can mediate the entrance into society by the learners. A third aspect involves rethinking knowledge and learning, and incorporating here the search for meaning within this growing knowledge society. Here we draw on Biesta (2009) who envisages the outcomes of education to include independent thinking and indeed an approach to society as emergent rather than fixed.

While teacher education development has many facets we aver that teacher agency, the dynamic competence of human beings, and here of teachers to respond to the needs in their environment, the classroom and school environment, is the keystone. We note here that while we consider this a competence inherent in every teacher, we acknowledge with Priestley et al., (2013), that agency is manifest in the interaction with the environment, and as such can be enabled or curtailed.



We would expect from teachers an honest appraisal of their own fears and lack of mastery, and then the realization that they have reasoning ability and resources within, and in addition have access to external resources through professional teacher organisations to increase their knowledge of particular topic areas and to build their confidence as problem solvers.

It is however, in our view, an insult to “test” teachers and principals as though a test can measure the work of the vocation, teaching. The approach to education, and particularly mathematical reasoning and problem solving requires a finer approach, where rather than to “fix” (through testing?), the appropriate verbs are to model, inspire, unfold and enable emergence.

### **The role of professional teacher organisations**

It has generally been the case that teachers have worked in isolated pockets and that an environment of competition rather than collaboration has existed in many schools. In a study on the implementation of the new Scottish curriculum, Priestley, et al., (2013) found more evidence of agency where there was a collaborative and horizontal structure rather than a hierarchical structure.

Professional mathematics teacher organizations have an important role to play in the creating an atmosphere of trust and through this trust affirm the agency, the dynamic and creative interaction with their environment, which includes interaction with the subject, mathematics. In both national and regional conferences the professional identity may be affirmed, and the worthwhile vocation of teaching elevated. The aim of professional teacher organisations such as the Association for Mathematics Education of South Africa (AMESA), together with the education departments, is to acknowledge the importance of the work of every teacher, and bring about a change from lack of confidence to a productive disposition where teachers are prepared to take on the challenges of learning and teaching, for example geometry, drawing on both internal and external resources, provided through various channels.

Professional teacher organisations are in a position to stimulate and facilitate discussion among teacher educators in order to oppose fragmentation and enhance coordination from within the community of teacher educators. In this way teacher educators can maintain agency and be proactive and adaptive to needs. Professional teacher organisations are also able to provide platforms for continued engagement between teacher educators and in-service teachers. It already does, and this should be acknowledged and supported, if necessary by some formal recognition for teachers who attend such workshops. Lastly professional organisations are able to provide a platform for teacher education to engage as a group with policy makers, government officials, textbook writers and “other role players” through planned focus group discussions of policy and research.

In conclusion, Pickering has used the metaphor the *dance of agency* (1995, p. 21), which he understood to take the form of a “dialectic of resistance and accommodation” (p. 22).

When applied to mathematics teacher education and mathematics education, this metaphor implies the movement of ideas, and a vibrant mathematical discourse within lecture halls, and within classrooms, and between teachers and students, in the interests of achieving a more creative, but at the same time a rigorous, education.

## REFERENCES

- Batra, P. (2009). Teacher Empowerment: The Education Entitlement-Social Transformation Traverse. *Education Dialogue*. Volume 6. No 2. Spring 2009.
- Biesta, G. (2009a). Good education in an age of measurement: On the need to reconnect with the question of purpose in education. *Educational Assessment and Evaluation*.
- Biesta, G. (2009b). Good Education: What it is and why we need it, Inaugural lecture. Stirling: The Stirling Institute of Education.
- Biesta, G.J.J & Tedder, M. (2006). How is agency possible? Towards an ecological understanding of agency-achievement. Working paper 5. Exeter: The Learning Lives project.
- Carl, A. E. (2009) *Teacher Empowerment Through Curriculum Development: Theory into Practice*. Cape Town: Juta Academic
- Colapietro, V. (2009). A Revised Portrait of Human Agency: A Critical Engagement with Hans Joas's Creative Appropriation of the Pragmatic Approach *European Journal of Pragmatism and American Philosophy*
- Department of Basic Education (2011). *Integrated Strategic Planning Framework for Teacher Education Development in South Africa*. Pretoria:DBE
- Department of Education, 1998, Norms and Standards for Educators, Technical Committee on the revision of Norms and Standards for Educators, Department of Education: Pretoria
- Griffiths, M. (2013). Imagination and Creativity in Teacher Education. *European Educational Research Journal*
- Goldthorpe, J. (1998) Rational Action Theory for Sociology. *The British Journal of Sociology*, Vol. 49, No. 2, pp. 167-192
- Gone, T., Miller, P. & Rappaport, J. (1997). Conceptual Self as Normatively Oriented: The Suitability of Past Personal Narrative for the Study of Cultural Identity
- Higgins, C. (2013). *The good life of teaching*. London: Wiley
- Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics*. Washington DC: National Academy Press.
- Pickering, A. (1995). *The mangle of practice: Time, agency, and science*. Chicago: University of Chicago Press.
- Priestley, M., Biesta, G.J.J. & Robinson, S. (2013). Teachers as agents of change: teacher agency and emerging models of curriculum. In M. Priestley & G.J.J. Biesta (Eds.), *Reinventing the curriculum: new trends in curriculum policy and practice*, London: Bloomsbury.
- Schoenfeld, A. H. (2007). What is mathematical proficiency and how can it be assessed? In A. H. Schoenfeld, (Ed.), *Assessing Mathematical Proficiency* (Vol. 53). MSRI Publications.
- Wagner, D. (2007). Students' Critical Awareness of Voice and Agency in Mathematics Classroom Discourse', *Mathematical Thinking and Learning*,9:1, 31 — 50
- Wylde, D. (2011). "We have knowledge we must share it" (<http://leadsa.co.za/?p=7483>).



## ERRORS AND MISCONCEPTIONS IN SOLVING QUADRATIC EQUATIONS BY COMPLETING A SQUARE

**Sello Makgakga**  
Mathematics Education

### **Abstract**

*This paper is based on the qualitative study that was conducted in five South African schools of Limpopo Province in Capricorn district. The main aim was to diagnose errors learners made in solving quadratic equations by completing a square and found the reasons why those errors occurred. The results revealed that learners in those schools experienced problems in solving quadratic equations by completing a square. Findings revealed that the teaching approaches used by teachers contributed towards learners' errors and misconceptions they possessed in solving quadratic equations by completing a square. Further, findings showed that learners were not given an opportunity to discuss the concepts and they did not have enough time to practice their work. Other reasons found in the five schools are – learners are afraid to be mocked at by their peers during the lessons, learners themselves not committed to their school work, some have parental role to play at home, and some teachers are not committed as they come to class late and not controlling their work. The recommendation is that in identifying learners' errors and misconceptions teachers should also diagnose those errors in order to understand learners' challenges.*

### **INTRODUCTION**

Mathematics resolves around conceptual dissemination and skills attainment and enhancement. This enables learners to acquire new mathematical skills and knowledge or concept to consolidate the old ones (Luneta, 2008). Teachers in schools use different teaching methods such as group work, exposition and explanatory, problem solving, practical work, direct instruction; this may be the application of constructivist approach in teaching and learning of mathematics. The key element in the approaches is for learners to use mathematical skills and knowledge confidently in solving mathematical problems. Learners are evaluated thereof to on their understanding of the concepts in mathematics and the quality of teaching, and this helps to reveal learners' errors and misconceptions (Borasi, 1994; Riccomini, 2005).

Mathematics teachers did not treat learners errors made in solving mathematical problems seriously. Instead, those errors were ignored, under the guise that mathematics was being undertaken by intelligent learners and those whose performance is on average were reflective of ‘general streams’. This issue of ignoring the errors and misconceptions of learners in mathematics also occurred in the five schools I used in the study. Teachers did not have courage to investigate learners’ problems experienced in solving mathematical problems, until I had a casual discussion on performance of learners’ with two teachers from different schools in the vicinity.

Arising from this casual discussion, it was concluded that there was a need to analyse concepts that seemed to pose challenges. I started by looking at simple concept like grade 11 quadratic equations and teachers identified to me that learners had a serious problems in solving quadratic equations by completing a square. The poor performance of learners motivated me to think more about the types of errors learners displayed and misconceptions they possessed. In subsequent discussion with the teachers, we decided to give learners a diagnostic test to understand those errors and misconceptions in solving quadratic equations by completing a square.

This paper describes the types of errors and misconceptions the grade 11 mathematics learners grapple with and why those errors displayed and how they arrived at such misconceptions in solving quadratic equations by completing a square. The study also discusses in conceptualisation and interpretation which lead learners to wrong answers. It also aimed to assist teachers to be more sensitive about learners’ errors and misconceptions in mathematics. This study will also assist learners to realise the types of errors they display in solving quadratic equations by completing a square and other mathematical concepts. They gained conceptual and procedural knowledge of the quadratic equations by completing a square.

## LITERATURE REVIEW

### Definition of errors and misconceptions

Hansen et al., 2006: 15-16 define errors and misconceptions ‘errors are mistakes learners make when solving problems that may be caused by carelessness, misinterpretation of symbols or text; lack relevant experience or knowledge related to that mathematical topic, learning objective, concept; lack of awareness or inability to check the answer given; or the results of misconceptions. Misconception is the misapplication of a rule, an over- or under-generalisation, or alternative conception of the situation’. Error discourse has since been discussed long time ago in the 17<sup>th</sup> century for instance, the work of Bacon in 1620 and also Pierce in 1887 cited in Luneta 2008. Learners make errors in some situations without realising them and if those errors are not recognised or dealt with, those errors can lead to more errors (Pickthorne, 1983).



Pickthorne (1983: 1285) also asserts that too often lecturers appear not fully realise the extend or the nature of their learners' confusions. They are unable to discover the reasons why those errors are made by the learners.

When we diagnose mathematical difficulties, we determine the areas of weaknesses a learner has, we study a specific errors the learner is frequently making, and we attempt to explain why those errors are being made (Troutsman & Alberto, 1982). Most of the teachers identify learners' errors but rarely diagnose them (Luneta, 2008). When analysing and diagnosing errors leaners displayed, we identify the root cause of those errors and how best they can be corrected in order to benefit both learners and teachers. Learners will not commit same errors and teachers will not teach same concept to the same learner. Errors and misconceptions in mathematics are mostly influencing learners' learning of mathematics and hence quadratic equations by completing a square, my focus area. Teachers' mistreatment of errors and misconceptions has exacerbated their effects and has retarded learning (Pickthorne, 1983).

### **Quadratic equations by completing a square**

This method of solving quadratic equations by completing a square is helpful as it was appropriately applied in finding the solution to the equations; learners were alerted to use this method appropriately to provide them with the correct answers. In completing a square according to Laridon et al. (2010), learners should ensure that the coefficient of  $x^2$  is 1 and if it greater or less than 1, they should divide by that coefficient,  $ax^2 + bx + c = 0$ . Learners should divide by  $a$  throughout before they could find the additive inverse of  $c$  both sides to have  $x^2 + \frac{b}{a} = -\frac{c}{a}$ . Learners can then add half the coefficient of  $x$  both sides before the equation could be factorised,  $x^2 + \frac{bx}{a} + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$ . The equation can then be factorised on the left hand side and be simplified on the right hand side (Pretorias et al., 2006).

### **Conceptual and procedural knowledge**

Conceptual knowledge is knowledge about the facts, concepts or principles upon which something is based (Microsoft Office Dictionary, 2001). Herbert and Lefevre (1986: 3-4) define conceptual knowledge as 'knowledge that is rich in relationship, that can be thought of as a connected web of knowledge, a network in which linking relationships are as prominent as the discrete pieces of information'. Such knowledge is described as that which is interconnected through relationships at various levels of abstraction (Confrey, 1990). It plays a more important role in the learning of mathematics than procedural one. It is essential for learners to have conceptual understanding, as in the absence of which they will ineffectively indulge in problem solving and follow wrong procedures to solve them (Center for Develop Enterprise, 2007).

Zemelman, Daniels, & Hyde (1998: 89-90), state that ‘without true understanding of the underlying concepts, serious problem [guaranteed] with learning other concepts’. Focusing on understanding mathematical ideas makes students ‘far more likely to study mathematics voluntarily and acquire further skills as they are needed.’

Teachers should know their learners’ mathematical thinking to be able to structure their teaching of new ideas to work with or correct those ways of thinking, thereby preventing learners from making errors (Sorensen, 2003). The way learners think about a concept depends on the cognitive structures learners have developed previously (Battista, 2001). Battista (ibid) also indicates that if learners cannot develop concepts by themselves, they will have a narrow understanding of those specific concepts, and will not be able to engage themselves in problem solving. Learners who do not have background knowledge in mathematics usually display numerous errors in solving mathematical problems, and this therefore results in most of learners grappling with quadratic equations by completing a square. Conceptual knowledge works hand in hand with procedural knowledge.

Procedural knowledge should be taught in mathematics to reinforce understanding of mathematical concepts. Procedural knowledge is the ability of learners to use the relevant procedures in solving mathematical problems by following the rules, methods and procedures in different representations (Kanyaliglu, Ipek, & Isik, 2003; Hiebert & Lefevre, 1986). Procedural knowledge is a particular type that learners display in solving problems and also in adhering to certain instructions when completing different tasks (Hiebert & Lefevre, 1986). Luneta (2008) asserts that procedural knowledge is specific to a particular task, and this implies that some procedures are not appropriate to solve certain mathematical problems. Learners may grasp relevant procedures but fail to use them correctly in solving mathematical problem (Siegler, 2003). Learners who lack this understanding are frequently using wrong procedures and this generates systematic patterns of errors in solving problems. Accordingly, teachers should not focus only on fact errors, but also on basic errors, especially when learners are making the same procedural errors (Gernett, 1992; National Research Council, 2002; Stein et al., 1997).

Riccomini (2005) explains instructional focus on facts as being problematic to teachers teaching parts of concepts or parts of procedural steps because teachers are trained to teach mathematics in terms of general concepts, and therefore this helps in addressing the problem of learners solving quadratic equations by completing a square. Riccomini also states that the teachers’ ability to recognise error patterns can be improved and that it might be possible to plan instructions that can help to alleviate problematic patterns in this concept. Measures to this effect might be pre-service programmes, professional development opportunities in mathematics, refining curriculum materials, and continued research in mathematics for learners with disabilities.



There is a relationship between procedural knowledge and conceptual knowledge. The correlation is shown when a learner is able to execute the procedures correctly and this displays a good grasp of conceptual knowledge (Siegler, 2003). If a learner has both procedural and conceptual knowledge, s/he can solve more complex problems of the same concept and this will help grade 11 learners solve quadratic equations by completing a square. Learners with conceptual understanding produce substantial gain in in both kinds of knowledge, but those with procedural understanding produce substantial gain in procedural knowledge but less in conceptual knowledge, which will ultimately impede learners' growth in mathematics. Nesher (1986) supports the view that if a learner can only be shown procedures of solving a particular problem without understanding the concept, it is very unlikely that such a learner would be in a position to solve more complex problems independently. If problems are difficult to solve, then learned procedures may not help and will need a learner to have conceptual knowledge to solve them Nesher (1986). Conceptual knowledge also provides and constraints those procedures to be followed in solving mathematical problems (Confrey, 1990).

## RESEARCH DESIGN

The study used a diagnostic test followed by focus group interviews to understand the reasons behind the errors and misconceptions learners had in solving quadratic equation by completing a square. It was conducted under qualitative research approach in which the information collected was analysed through description and not statistically (Rule & John, 2011). The study had been conducted in five schools of same circuit which comprises of 11 secondary schools. The focus of the study was on diagnosing errors and misconceptions learners experienced in solving grade 11 quadratic equations by completing a square. Sixty five scripts were sampled from five schools, in which 15 scripts were randomly sampled from each school and 10 learners participated in the focus group interviews which gave a total of 50 learners. Each group of learners were represented by five female learners and five male learners to observe gender equality. The reason for these participants was to find out the root causes and reason why those errors and misconceptions occurred in solving those equations by completing a square.

## Instrument

The instruments used in this study were the diagnostic test and interview schedule. The diagnostic test assisted in identifying and diagnosing the types of errors and misconceptions learners had in the five schools in solving quadratic equations by completing a square (Salvia & Yssedykes, 2004). The interview schedule helped me in understand the reasons why learners displayed those errors and misconceptions possessed.

The collected data through diagnostic test and focus group interviews attempted to respond to the following objectives of the study:

1. To identify the types of errors learners displayed in solving quadratic equations by completing a square and the misconceptions possessed and analyse them.
2. To diagnose those errors and misconceptions learners had and how well they could be addressed.

## FINDINGS AND DISCUSSIONS

The findings and discussions are divided into two components in which the first one dwells on the types of errors and misconceptions identified in solving quadratic equations by completing a square. The second component dwells on the reasons why those errors and misconceptions occurred by using a focus group interview. Furthermore, some reasons why those errors were displayed by those learners were found and were benchmarked against the data collected from the diagnostic test.

### Component 1: Types of errors and misconception

The first component of the analysis of learners' scripts in solving quadratic equations by completing a square was to identify the errors and misconceptions learners made. The collected data was analysed by identifying the common errors learners displayed in solving equations by completing a square. Those common errors were characterised by conceptual errors and procedural errors. Most of the common errors found were dividing by the coefficient of  $x^2$  if the equation was greater than 1 or less than zero. Moreover, some of the common errors were the ones of simplifying  $(2 \times \frac{1}{2})^2$  from the equation,  $x^2 - 2x - 1 = 0$ . Most of them got the answer as 2 and pursued them by the focus groups to understand why they solved the problem in that way. Other learners did not find the additive inverse of a constant -1 before completing a square, which was also wrong for them to solve the equation in that fashion. More errors were found when some of them were failed to factorise the equation after completing a square which revealed that learners lacked knowledge of factorisation. Others rewritten their equations in the form of  $ax^2 + bx + c = 0$  and then factorised instead of factorising the equation without writing it in standard form. Some of them found the additive inverse of the equation but only completed a square on the left hand side and failed to do it on the right hand side. Another common error was that some of the learners had failed completely to attempt to complete a square in solving quadratic equations, such as dividing the coefficient of  $x^2$  if it was greater or less than 1. Laridon et al., (2011) advise that when adding half the coefficient of  $x$ , learners should ensure that the coefficient of  $x^2$  is 1. They also further stated half the square of the coefficient of  $x$  should be added both on the left hand side and the right hand side. This view was supported by Zemelman et al., (1998) that learners without true understanding of the underlying concepts guarantee serious problems in learning other concepts. This is what happened to these learners as this reveal that most of



them were unable to solve quadratic equations by finding factors and this had continued on to this one of completing a square. Siegler (2003) also asserted that a learner without a good understanding of a concept result in using procedures of solving problems inappropriately. Most of these learners did not know how to solve the equations and showed that specific knowledge of procedures was not acquired by those learners. Luneta (2008) supports that procedural knowledge is specific to a particular task. If a learner does not understand the concept, it would be difficult for him/her to articulate the procedures to solve problems based on that particular concept (Battista, 2001).

### **Component 2: Diagnosing errors and misconceptions**

In diagnosing errors and misconceptions, I understood the reasons why the learners had those errors and misconceptions in solving quadratic equations by completing a square. The reasons behind those errors and misconceptions were found by using focus group interviews. In focus groups, some indicated that this concept is challenging to them as they compared with factorisation and using quadratic formula. In the discussion, learners revealed that some of their teachers don't give them an opportunity to participate and only the fast learners were always given a platform. In some schools learners were unable to participate as their teachers praised those who gave correct answers. Those learners were again unable to participate as they were afraid of being mocked at by their fellow learners. Learners also blamed themselves for not being committed to their school work due to their laziness, some mentioned that they had parental role to play at their homes. It was also revealed that the approaches teachers used to teach mathematical concepts were difficult to understand. They said some of the teachers were fast and don't give learners chance to engage with them. It was also indicated that other teachers were not committed to their work, like coming to class late and not controlling their work. From the focus group interviews, most of the learners were not given an opportunity to discuss the concepts and some indicated that they did not have enough time to practice their work.

### **CONCLUSION REMARK**

It is imperative for teachers to teach mathematics using learners' errors and misconceptions as this will guide them on what learners grapple with. They will be able to use multiple strategies or approaches to teach mathematical concepts to cater all the learners including the slow ones. In this study learners were unable to divide by the coefficient of  $x^2$  if the equation was greater than 1 and less than zero, multiplying by half the coefficient of  $x$  and also additive inverses of any constant given. Some reasons which contributed to these difficulties were: learners' laziness, learners' lack of participation, teachers' teaching participation, teachers' commitments and conduct towards learners; and peers' behaviour in the classroom. Teachers are not supposed to identify learners' errors and misconceptions but they should also diagnose those errors in order to understand learners' challenges.

## REFERENCES

- Battista, M. T. (2001). A Research-based Perspective on Teaching School Geometry. In J. Brophy (Ed.), *Subject-specific instructional methods and activities, advances in research on teaching*. (Vol. 8), (pp. 145-185). Elsevier Science.
- Borasi, P. (1994). The New Organ or true directions concerning the interpretation of nature. [www.constitution.org/bacon/nov\\_org.htm](http://www.constitution.org/bacon/nov_org.htm). Also in the Great Instruction/Novum organum. Modern Library. 1955 (1620).
- Centre for Development Enterprise (CDE, 2007). *Doubling for Growth: Addressing the math and science challenges in South Africa*. Centre for Development Enterprise.
- Confrey, J. (1990). A Review of the Research on Student Conceptions in Mathematics, Science and Programming. In C. B. Carden (Ed), *Review of Research in Education*. Washington: American Educational Research Association.
- Garnett, K. (1992). Developing Fluency with Basic Number Facts: Interventions for Students with Learning disabilities. *Learning Disability Research and Practice*, 7, 210-216.
- Hiebert, J., & Lefevre, P. (1986). Conceptual and Procedural Knowledge in Mathematics: An Introductory Analysis. In J. Hiebert (Ed), *Conceptual and Procedural Knowledge: The Case of Mathematics*. pp. 1-27. Hillsdale, NJ: Erlbaum.
- Kanyalioglu, A. C., Ipek, A. S., & Isik, A. (2003). On the teaching linear algebra at University level: The role of visualising in the teaching vector space. *Journal of Korea Society of Mathematics Education Series D. Research in Mathematics Education*, 7(1), 59-67.
- Laridon, P., Barnes, H., Jawrek, A., Pike, M., Myburgh, M., Houghton, R. R., Scheber, J., Segabi, M., & Wilson, H. (2010). *Classroom Mathematics Grade 11*. Sandton: Heinemann Publishers.
- Luneta, K. (2008). The professional development model of evaluating and enhancing. *In the International council of the Education of the Teachers: Year book 2008*.
- Microsoft Office Dictionary 2001*
- National Research Council (2002). Helping children learn mathematics. *Mathematics Learning Study committee*, J. Kilpatrick & J. Swafford, (Eds), Centre for Education. Division of Behavioral and Social Sciences and Education. Washington, DC: National Academy Press.
- Nesher, P. (1986). Are Mathematical Understanding and Algorithmic Performance Related? *For the Learning of Mathematics*, 6(3), 2-9.
- Pickthorne, B. (1983). Error factors: A missing link between cognitive science and classroom practice. *Instruction Science* 11: 281-312.
- Pretorius, J., Potgieter, R., & Ladewig, W. (2006). *Mathematics Plus, Grade 11*. Cape Town, Oxford University Press.
- Riccomini, P. J. (2005). Identification and remediation of systematic error patterns in subtraction. *Learning disability Quarterly*, 28(3), 233-242.
- Salvia, J. & Ysseldykes, J. E. (2004). *Assessment* (9<sup>th</sup> ed) Boston: Houghton Mufflin Company.
- Siegler, R. S. (2003). *Implantation of Cognitive Science Research for Mathematics Education*. Carnegie Mellon University. In Kilpatrick.
- Sorensen, A. (2003). *Effective Teaching high School Mathematics*.



- Stein, M., Silbert, J. & Carnine, D. (1997). *Designing effective mathematics instruction: a direct instructional approach (3<sup>rd</sup> ed.)*, Columbus OH: Merrill.
- Troutsman, A.C. & Alberto, P. A. (1982). *Applied Behavior analysis for Teachers: Influencing student Performance*. Columbus, OH: Merril Publishing.
- Zemelman, S., Daniels, H., Hyde, A. (1998). *Best practice: New standards for teaching and learning in American's schools (2<sup>nd</sup> ed.)*. Portsmouth, NH: Heinemann.

## PREPARING TO TEACH GRADE 12 FINANCIAL MATHEMATICS: A CASE STUDY OF STUDENT TEACHERS' LEARNING EXPERIENCES

Makonye J. P., Weitz M. and Parshotam B.

*This paper focuses on identifying the errors that second year student teachers, majoring in mathematics, make and on understanding the misconceptions that they have with regard to grade 12 financial mathematics. Data from seventy students' responses to grade 12 financial mathematics examination tasks were collected and analysed, after which twelve of these students were interviewed about the mathematical reasoning that they used. Our findings indicate that students' errors and misconceptions on this topic are conceptual, procedural and linguistic. We argue that it is critical for students to be familiar with the geometric sequence - the map of financial mathematics. Also, we argue that financial mathematics terms need to be thoroughly taught because many student errors on this topic also stem from misunderstandings of terminology.*

### INTRODUCTION

This article is about mathematics major student teachers' errors and misconceptions in the area of financial mathematics. Using Davis' (1984) frames as knowledge representation structures, and Tall and Vinner's (1981) concept definition and concept image notions, grade 12 financial mathematics tasks were given to second year mathematics major students. This was done to determine the errors that students still held on this topic even though they were now at university. Such as study is seen by the authors of this paper, who are the lecturers of these students as a methodology of informing good preparation for teaching our student teachers.

Financial mathematics is not only an interesting mathematics topic in its own right, but it is an aspect of mathematics that has applications in daily life, for everyone. Yet many people do not understand even the basics of financial mathematics. Mathematics major student teachers who do not understand this topic often fail one of their maths courses. In this article we argue that exploring student teachers' errors and misconceptions in financial mathematics can help researchers and teacher educators to identify key epistemological factors which could help stakeholders to better handle the topic.

Exploring student's epistemological difficulties through error analysis and diagnosis is an essential component of quality teaching (Makonye & Luneta, 2013; Nesher, 1987). It is important because it informs researchers, teachers and students themselves about the difficulties on specific mathematical objects that learners encounter.

Errors provide educators insight in to learners' current knowledge that can be used as a resource for teaching (Gallagher, 2004; Borasi, 1994). Errors can reveal to teachers exactly what learners think about certain mathematical work. Shulman (1986), argued that the teacher's knowledge of the likely errors and misconceptions that learners are prone to, as individuals or as groups, when they encounter particular mathematical concepts constitute Pedagogical Content Knowledge (PCK). Shulman concluded that educators would not be effective teachers if they were not knowledgeable of the errors and misconceptions learners usually held on particular concepts. A teacher who is aware of the likely misconceptions learners might have on a class of mathematics concepts is empowered. He/she is likely to be on the lookout for them. Also such a teacher can hypothesise situations that induce learners to elicit those errors; what Smith et al. (1993) referred to as 'confronting students with their errors'. Once such learners voice their misconceptions in the open, educators can devise pedagogic approaches to help learners see that their point of view is in fact not good enough. For these reasons, it is important for mathematics teacher educators and the student teachers with whom they work to determine the errors and misconceptions that cause difficulties for these students.

Over the last three decades, many papers have been written on the misconceptions that school children harbour when learning arithmetic, algebra, geometry, statistics and probability to name a few. These papers report that the errors exhibited, whether shared or idiosyncratic follow carefully reasoned patterns and are quite predictable, provided one have understood them. However there have been very few papers written to diagnose the errors and misconceptions that the student teachers and teachers themselves have in the mathematics content that they teach children. Such teacher errors and misconceptions are quite problematic because the teachers inadvertently pass on wrong ideas to the learners, which they then take up as knowledge. One important area in which teachers' exhibit errors and misconceptions is financial mathematics.

### **AIM**

The research aims to explore the errors that mathematics major students have on grade 12 financial mathematics. It also aims to explore what frames of reasoning induce them to have those errors.

This paper focuses on identifying the errors that second year mathematics major students have on grade 12 financial mathematics. It aims investigate the reasons why students hold on to these errors on grade 12 financial mathematics.

## SIGNIFICANCE OF THE RESEARCH

Since the onset of democracy, the average performance of South African students on periodic international comparative mathematics tests have been consistently under expectation (see Howie, 2003; Reddy, Winnar, Visser, Arends, Mthethwa, Juan, Rogers, Feza-Piyose & Prinsloo, 2013).). One would suspect that such attainment on the part of learners might not be wholly their fault. It might be that some of the teachers who teach these students are not necessarily more-knowledgeable others in Vygotskian terms. Indeed mathematics teachers who mark matriculation examination scripts were recently themselves requested to write their learners' grade 12 mathematics examinations. The results were not surprising in that about 60% of the teachers could not pass the examinations written by their own learners. At our own school of education fourth year students about to graduate also dismally failed in a similar examination. This shows that the mathematics content knowledge for our mathematics teachers cannot be taken for granted. It is imperative that there be research which explores the content knowledge of mathematics student teachers. This is important in that prospective teachers' knowledge deficit can be identified, understood so that these can be rectified before the students graduate. Since student teachers bring to higher education the mathematical errors they hold from high school, a research which brings to light such errors in important one as it helps to ensure that teachers who graduate do not have embarrassing deficits in the content knowledge of the mathematics they will teach.

## THEORETICAL FRAMEWORK

The research is studied through constructivism and particularly through Davis' (1984) work on frames that learners construct and use to learn and do mathematics. Constructivists argue that learners are not explicitly taught the errors and misconceptions they have (see for instance Confrey & Kazak, 2006; Davis, 1984; Smith et al., 1993). Student errors and misconceptions are their current knowledge which they use to connect to new knowledge. If new knowledge is connected to current knowledge that is incorrect unfortunately another error will occur. Also even if current knowledge is correct, errors can result of the current correct knowledge is being connected to new knowledge. The new knowledge might not be portable with current knowledge and the learner is mistaken that it is. Definitely new knowledge cannot sit in a vacuum, it holds on to something however tenuous a learner previously learnt.

Some education theories such as behaviourism (Skinner, Pavlov, & Thorndike, as cited in Todes, 2002; McLeod, 2007) viewed errors as undesirable in the learning process. Their stance was that errors must be punished and weeded out so that they become extinct. On the other end of the spectrum, constructivists view errors as most useful resources in teaching and learning.



They see errors as learners' attempt to construct meaning in a learning context. To constructivists, once a learner shows an error, the teacher must be strategic about it. He/she must refrain from immediately supplying a correct answer as it would be missing an opportunity. Rather, the teacher helps the learner to reconsider his answer by requesting the learner to compare his answer with that of one or two other colleagues. That way, that learner encounters peer induced cognitive conflict that could help her to realize her error.

So learners build misconceptions by themselves as they strive to interpret new experiences and give meaning to them. According to Davis (1984), student's errors are not random; on the contrary, they turn out to be very regular and systematic. They have specificity and determinism and it is often possible to predict exactly which wrong answer is most likely to be given by a particular student. So systematic wrong answers given by a student often provide clues as to how the student is thinking about a certain class of mathematical problems.

This research hinges on frames (Davis, 1984) and concept image (Tall and Vinner, 1981). Such representations and images are explained in that learners and students do not absorb knowledge exactly in the form they are taught by the teachers. Rather they cognise it and try to fit it in what they already know. They accommodate and assimilate new knowledge to their previous experience. Previous experience provided students with frames for processing knowledge. According to Davis, students use the frame they have to interpret and process new knowledge.

Assimilating and accommodating new knowledge in old frames might be problematic as the frames might be overstretched to generalise on new platforms they do not apply. Such causes an error, a wrong answer due to an inappropriate framework being used to interpret knowledge.

## **METHODOLOGY**

The research was a qualitative design. About seventy, second year mathematics major students of both sexes were given Grade 12 financial mathematics examinations tasks from previous years. The tasks were collected and analysed for errors. After the tasks were analysed, some student teachers were also interviewed to find out what thinking they held made them to produce wrong answers. 12 students chosen for interview constituted a stratified sample of capability and sex.

Students' response to tasks were first analysed under the categories of correct, partially correct and incorrect as well as not attempted. In particular students' scripts were analysed in greater detail about the errors they made. The errors were analysed under the categories of conceptual, procedural, arbitrary and careless. However in the analysis, the researchers were open as to the new errors that did not fit into these categories.

Also the errors were discussed in the lens of how students constructed knowledge; knowledge representation structures (Davis, 1984) as well as concept images (Davis, 1984). The interviewees will be transcribed so that the main themes that cause students to have errors can emerge.

## DISCUSSIONS AND FINDINGS

We argue that students have errors in financial mathematics because they are not familiar with the mathematical concepts underlying mathematics and connections between them. Such concepts are the percentages, time, geometric sequence including the  $n$ th term and the sum of the geometric sequence. While some students understand these underlying concepts, they have problems in the application of the series to a practical context such as financial mathematics. Students also have errors and misconceptions because they do not understand financial mathematics terminology, such as simple interest, compound interest, nominal interest rate, effective interest rate, balance, principal, balance, loan, deposit, annuity and so on. Students also have a great deal of errors because they try to answer the questions by blindly substituting values in financial formula.

Thus students' errors in financial mathematics are mainly conceptual. Students have problems on this topic because they want to work it out procedurally without understanding the mathematics of the geometric sequence on which financial mathematics depends. Given this scenario it is important that once students' errors and misconceptions are identified, then teaching must focus on those errors and misconceptions so that they can be resolved. One way to help students resolve their errors and misconceptions is through introducing cognitive conflict. We argue that teaching that does not start with diagnosing students errors and misconceptions in a certain topic is one of the reasons why teaching can be a sterile activity, as it does not connect with the knowledge representation structures that mathematics students have. Helping student teachers to relieve the misconceptions they have in mathematics and financial mathematics goes a long way in ensuring that misconceptions in mathematics are not passed from generation to generation.

In distinguishing between student teachers' errors and misconceptions in financial mathematics it is our purpose to highlight the connections between students' errors and misconceptions by pointing out that the errors that students' exhibit in financial mathematics are an outgrowth of consistent and often understandable misconceptions; which are the underlying cognitive representation structures responsible for errors. Besides providing a map of how students' unseen misconceptions result in errors, we assess the extent to which underlying misconceptions lead to the difficulties students face in learning mathematics.



The article is structured as follows: After giving an overview of literature on errors and misconceptions in mathematics, we outline the purpose of the research and the research questions. Next we provide the theoretical framework for the research and the research methodology. Finally, in the last two sections we analyse the data on student errors in financial mathematics tasks and the data from interviews with selected students and conclude with a discussion of our findings.

The study noted that students do not understand the meaning of financial mathematics terms such as compound interest, effective interest rate or annuity. Also students' misunderstanding of the geometric sequence and its inherent application to financial mathematics resulted in many errors. Blind substitution into financial mathematics formulae was common.

## REFERENCES

- Borasi, L. (1994). Capitalizing on Errors as "Springboards for Inquiry": A Teaching Experiment. *Journal for Research in Mathematics Education*, 25(2), 166-208.
- Confrey, J., & Kazak, S. (2006). A thirty-year reflection on constructivism in mathematics education in PME. In P. Boero and A. Gutierrez (Eds.), *Handbook of research on the psychology of mathematics education: past, present and future* (pp.305-345). Rotterdam: Sense Publishers.
- Davis, R. B. (1984). *Learning mathematics: The cognitive science approach to mathematics education*. Norwood, N.J.: Ablex Publishing Corporation.
- Gallagher D 2004. The importance of constructivism and constructive pedagogy for disability studies in education. *Disability studies quarterly*, 24(2),
- Howie, S. (2001). *Mathematics and science performance in grade 8 in South Africa 1998/99: TIMMS-R 1999 South Africa*. Pretoria: Human Sciences Research Council.
- Makonye, J. P., & Luneta, K. (2013). Learner Mathematical Errors in Introductory Differential Calculus tasks: A Study of Misconceptions in the Senior School Certificate Examinations. *Education as Change*, 18(1), 119-136.
- McLeod, S. A. (2007). B.F. Skinner: Operant conditioning—simply psychology. Retrieved from <http://www.simplypsychology.org/operant-conditioning.html>
- Nesher, P. (1987). Towards an instructional theory: The role of learners' misconception for the learning of mathematics. *For the Learning of Mathematics*, 7(3), 33-39.
- Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14.
- Smith, J. P., diSessa S. A., & Roschelle, J. (1993). Misconceptions reconceived: A constructivist analysis of knowledge in transition. *The journal of learning sciences*, 3(2), 115-163.
- Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12(2), 151-169.
- Todes, D. P. (2002). *Pavlov's physiology factory*. Baltimore, M.D.: Johns Hopkins University Press
- Reddy, V., Winnaar, L., Visser, M., Arends, F., Mthethwa, M., Juan, A., Rogers, S., Feza-Piyose, N. & Prinsloo, C.H. (2013). *Highlights of TIMMS 2013, South Africa: Towards equity and excellence*. Pretoria: Human Sciences Research Council.

## MANAGING AND LEADING MATHEMATICS IMPROVEMENT IN FOUNDATION PHASE

Shauna Pamplin and Nicky Roberts

Capricorn Primary School, University of Witwatersrand School of Education

*This paper draws on our attempts to shift performance of mathematics at Foundation phase level in a township primary school over two years, as part of a three year research and development project. It is written from the perspective of the deputy principal of a primary school (the first author) in a challenging social context. While she does not teach directly in classrooms; she has been instrumental in shifting the way mathematics is taught in this school. The paper reports on the deputy principal's roles in procuring and managing mathematics related equipment; making mathematics visible throughout the school environment; involving parents; interventions for supporting under-performing learners and extending learners who are meeting expectations; and motivating and encouraging teachers to undertake professional development to improve their own mathematics.*

### INTRODUCTION

Shifting mathematics learning in a school is difficult and involves effort from a range of different role players in a schools community: school leadership; teachers; learners and parents. In this primary school we are seeing some positive shifts at Grade 3 level with improvements in both the Annual National Assessment results, and the Western Cape Education Department systemic tests. These shifts, we believe, are a combination of various efforts; with most of the acknowledgement for improvements vesting with the teachers who have each worked hard to better support their learners. At the same time we are aware of the role that school managers can play in supporting teachers in these efforts. As such we think it worth sharing the role of the first author, in her capacity as deputy principal and reporting on what she has done with regarding to leading and managing mathematics at this Foundation Phase level.

### Managing and leading mathematics improvements

In this paper we report on the deputy principal's efforts in several spheres of management and leadership for mathematics teaching and learning.

We focus first on procuring and managing mathematics related equipment in a context where teachers do not have much equipment; may not know how to use the equipment that is available; and may resist using equipment as this creates excitement and ill-discipline amongst learners. Here the deputy principal's experience of raising funding and managing the collection of recycled materials for use in mathematics may be of interest.

Secondly we provide a photographic tour of the school to display how the deputy principal has made mathematics visible throughout the school environment. We include descriptions of how this was made possible through innovative management of resources and sponsorship relationships.

Thirdly we report on the design and functioning of a dedicated mathematics hub for Foundation phase mathematics. We explain the rationale for this space, what equipment it includes and how this has been timetabled for mathematics teaching use in this school. This leads us into how the hub is being used in the afternoons for weekly mathematics clubs for grade 2 learners. The focus here is on games and fun mathematics to get children fluent and playful with mental mathematics. The school has recently become a full service school, and as such has the benefit of learning support staff. The deputy principal's role in overseeing these support interventions is therefore commented on. A recent focus has been involving parents, and we report on this engagement process which is in its infancy.

Finally we reflect on what has been done to motivate and support teaching staff in relation to mathematics. We include some description of how the integrated quality management systems function in the school as well as types of professional development offerings which teachers have been encouraged to participate in. In addition, we give insight into how the deputy principal created interest improving teachers mathematical knowledge by publically revealing how own insecurities relating to mathematics and undertaking professional development herself.

## CONCLUSION

There is much more which is still to be done to further improve mathematics in this school. There remain problems with learner discipline in the classrooms. Ensuring that all teachers are willing and confident to try new approaches; and balancing the need to “cover the curriculum” while ensuring children make sense of mathematics are areas we are trying to work on. We hope that this short paper provides some possibilities for how a school manager could approach adopting a proactive role in curriculum leadership for mathematics at Foundation Phase. What is being done is far from perfect, and much that has been implemented may not be feasible or successful in other primary school contexts. Nevertheless we feel it worth sharing our efforts, and are optimistic about the improvements have started to see in both learner results and teacher professionalism.

## ACKNOWLEDGEMENTS

This paper was made possible as part of Focus on Primary Maths project. This project is funded by ApexHi and administered by Tshikululu Social Investments.

## TEACHING WITH A ‘CRITICAL FEATURE’ OF AN ‘OBJECT OF LEARNING’ IN FOCUS

Vasen Pillay, Mampotse Shirley Ramasia & Taona Nyungu

*Four teachers and a researcher participated in a learning study. In this paper we provide a brief overview of learning study and then describe the problem we engaged with during our study. We focus on the second iteration of the lesson as it was central to the emergence of the critical feature. We then move to the fourth lesson since this was the only lesson where the critical feature for the object of learning was in focus for the teacher. We also reflect briefly on the learning gains made by the learners as well as on shifts in teachers’ thinking as a result of participating in this study.*

### INTRODUCTION

Improving something as complex and culturally embedded as teaching requires the efforts of all the players, including students, parents, and politicians. **But teachers must be the primary driving force behind change.** They are best positioned to understand the problems that students face and to generate possible solutions. (Stigler & Hiebert, 1999, p. 135)

In this paper we describe how through participation in a learning study we began to reflect critically on our lessons and started to become the ‘driving force behind change’ as alluded to by Stigler and Hiebert (ibid.), and this with respect to our teaching practice. Mampotse and Taona have been teaching secondary school mathematics for many years and find that learners have difficulty in understanding the notion of function. More specifically, they found that Grade 10 learners were unable to distinguish between different classes of functions given its algebraic representation e.g. which function represents a parabola:  $y = 2^x$  or  $y = x^2$ ?

In this paper we illustrate this difficulty and engage in discussion of how the teachers gradually shifted their practice to assist their learners in overcoming this difficulty. We commence this paper by providing a brief overview of what a learning study entails and in doing this we explain the concepts *object of learning* and *critical feature*.

### LEARNING STUDY – A BRIEF OVERVIEW

Marton and Pang (2006) characterise a learning study as a group of between two and six teachers working together to find a way of making it possible for learners to “appropriate a specific object of learning”(p. 194). They identify the object of learning<sup>11</sup> as “a specific insight, skill, or capability that the students are expected to develop during a lesson or during a limited sequence of lessons” (Marton & Pang, 2006, p. 194).



In other words, it can be those ‘hard to teach’ or ‘hot spot’ topics and concepts. Once the object of learning has been identified, the implementation of learning study follows a cyclical process. The process is considered cyclical because a lesson is collaboratively planned, it is taught by one teacher in the group and the other teachers observe the lesson. At the end of the lesson the teachers discuss the lesson and make suggestions for possible changes to the planned lesson. The lesson is revised and taught by the next teacher in the learning study group<sup>1</sup>. This process is repeated until all teachers in the group have had a turn to teach. Each of the lessons commences with a pre-test and at the end of the lesson the learners are required to write a post-test. This form of testing provides one possible way in which to track learning gains, if any, as a result of the lesson taught.

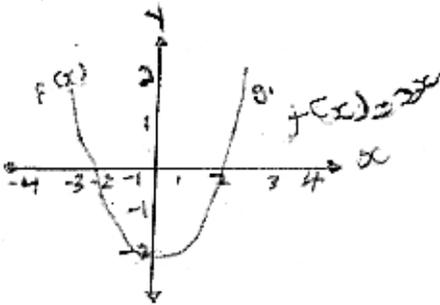
This cyclical process commences with the teachers choosing a specific object of learning that is central to the curriculum and that is known to cause difficulty for learners. Once the object of learning is identified the group commences with the planning of the lesson(s) with “a special focus on making it possible for the students to appropriate the object of learning” (Marton & Pang, 2006, p. 195). In their planning of the lesson the teachers in the learning study group have as resources their own experiences and previous research. In the study being reported on in this paper, the teachers also had a researcher as a resource. During the planning of the lesson the teachers focus on how they will vary some aspects of the object of learning while others remain invariant during the sequence of the lesson. In the case of this study, what varies and what is invariant is derived from the principles inherent in variation theory<sup>2</sup>.

The purpose of the iterative process is to provide opportunity for the emergence of a critical feature. It is important to note that a critical feature is not the same as the difficulty learners are experiencing in relation to the object of learning. Instead, it is the particular feature of the object of learning that the learners must be able to discern in order to experience the object of learning in a certain way. The critical feature cannot be drawn from the mathematics alone; it has to emerge in relation to the learners being taught. For example, the critical feature that emerged through this study was that in order for the learners to distinguish between the different classes of functions given its algebraic representation, their attention had to be drawn to the highest power of the independent variable.

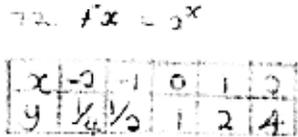
### OUR PROBLEM

Although we taught the section on functions to our Grade 10 learners during the second term they performed poorly on questions related to the function concept when tested in the mid-year examination. For the purpose of this paper we will focus on our learners' responses to the question<sup>3</sup>: In you answer script draw a neat graph of the equation  $f(x) = 2^x$ , show intercepts if any. The following extracts are typical learner responses to the above question:

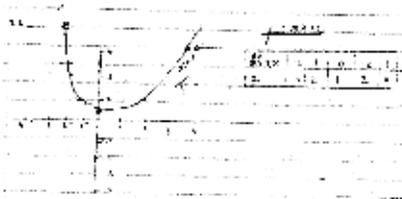
Response 1



Response 2



Response 3



Response 4

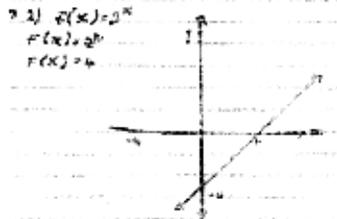


Figure 1: (Mis) recognition between multiple representations of a function

In Figure 1, responses 1, 2 and 3 suggest that the learners associated the function defined by  $f(x) = 2^x$  with the quadratic function. They possibly (mis)recognised the structure and characteristics between  $2^x$  and  $x^2$ .

In response 1, the learner did not show the derivation of the coordinates but drew a parabola with x-intercepts  $(-2; 0)$  and  $(2; 0)$  and y-intercept  $(0; -2)$ .



Interestingly, in response 2 learners were able to perform the correct arithmetic calculations after substituting various values of  $x$  into the given equation and represented these values correctly in a table, but plotted some of the points (where  $x$  is negative) incorrectly to form the shape of a parabola. In responses 1 and 2, it seems that the learners saw  $x^2$  and  $2^x$  as representing the same function. In response 4, the learners associated  $2^x$  with a linear function which has a positive gradient and it is unclear as to how they arrived at the  $x$ -intercept  $(4; 0)$  and  $y$ -intercept  $(0; -4)$ . In response 3, the learners signalled that they are interpreting  $f(x) = 2^x$  as being linear by writing the general equation of a linear function  $y = mx + c$ . Furthermore, in completing the table of values the learners changed  $2^x$  to  $2x$ , which is then written in the cell to indicate  $x$ -values in the table of values. The rule used to relate input values with their respective output values is not clear and does not satisfy the equation defined by  $y = 2x$ . However, after representing the coordinates as a table of values the learners proceeded to draw a quadratic function. One can again assume that for these learners as well,  $f(x) = 2^x$  represents the graph of a parabola.

Indeed there are indications that there are deeper difficulties for learners. Relatively simple algebraic forms appear to have little meaning for these learners. This raises some questions that are perhaps prior to multiple representations of functions: do learners see  $2^x$ ;  $x^2$ ; and  $2x$  as one and the same thing? What about  $\frac{x}{2}$  or  $\frac{2}{x}$ ? Do they understand the relationship between the variable and the constant in each instance? Do the learners recognise the mathematical operations implied in each of these instances and are they able to evaluate each of the expressions for a particular value of  $x$ ? This is our point of departure as we start planning the first lesson of the learning study. The object of learning being: to enhance learners' ability to differentiate between the linear, quadratic, hyperbolic and exponential function given its algebraic representation. The learning study reported on in this paper had four iterations. For the purpose of this paper we will only focus on the second and fourth lessons since these two lessons are the critical lessons in this learning study cycle. The critical feature only emerged after the second lesson and was only brought into focus during the fourth lesson.

## LESSON 2

To provide the context for lesson 2, we briefly describe what transpired in lesson 1. In lesson 1 the teacher worked across different classes of functions defined by:  $y = 2x$ ;  $y = x^2$ ;  $y = \frac{1}{2}x$ ;  $y = \frac{x}{2}$ ;  $y = 2^x$  and  $y = \frac{2}{x}$ . In working across these functions he focused on substituting values for the input variable, performing the required mathematical manipulations as per the given equation to obtain the corresponding output value. The teacher then represented the input and its corresponding output value in a table. These examples constituted the range of examples that was used in each of the four lessons.

Absent from this lesson was the graphical representation of the functions. This absence was included in lesson 2. Lesson 2 commenced with Mampotse displaying the graphs of planned examples and then learners were required to identify the type of function represented and then give the algebraic form. As the Mampotse progressed from one example to another the learners' difficulty in correctly identifying the type of function and providing its algebraic form became obvious.

During the post-lesson discussion the researcher, Vasen, posed the following question to the teachers: what makes  $y = 2x$  linear? Mathematically speaking  $y = 2x$  is not linear simply because it is a polynomial function of the first degree, it is linear because it has a constant rate of change and  $y = x^2$  is quadratic because the rate of change of the rate of change is a constant and not that it is a polynomial function of the second degree. In terms of relating the syntax of the algebraic representation of a function to its graph, focusing on the value of the exponent of the independent variable provides some criteria for learners to be able to identify the class of function being represented by a given equation.

#### LESSON 4

To set the scene for lesson 4 we briefly explain what happened in lesson 3. In planning for lesson 3, the emphasis was on bringing the degree of the equation into focus. Although this was the focus of lesson 3, the critical feature was not in focus for the teacher as she went about teaching the lesson. The teacher focused on substituting input values to obtain output values and then represented these values in a table.

In lesson 4 the Taona introduced two functions in their algebraic form (e.g.  $y = 2x$  and  $y = \frac{1}{2}x$ ) and asked learners to compare the equations and say what is the same and what is different. It is through this process that he focused his learners' attention on the power of  $x$ . This was the first lesson where learners were provided with some tools by which to identify the class of function being represented by a particular equation.

#### LEARNING GAINS

After the emergence of the critical feature, Mampotse went back and re-taught the lesson to her own learners. Her learners were the learners that were taught in the first lesson of this learning study cycle. This now meant that two of the four groups of learners were exposed to the critical feature that emerged during the post-lesson 2 discussion. A few weeks after the last lesson was taught all the learners that participated in this study were required to write a delayed post-test. In this section we present data from this test.



Consider the following sets of bar charts (Figures 2 and 3) which illustrate the learners' performance in the post-test specifically for questions that required learners to identify and name a function given its equation. The learners who were taught in the first lesson in the learning study cycle are referred to as group 1 learners, the learners who were taught in the second lesson are referred to as group 2 learners, etc.

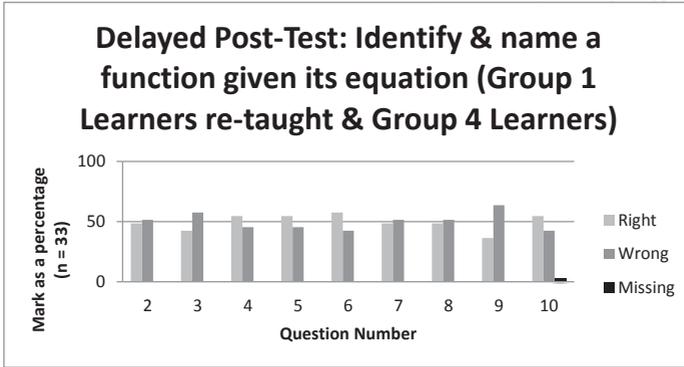


Figure 2: Post-test: identifying a function given its equation – Group 1 and Group 4 learners combined

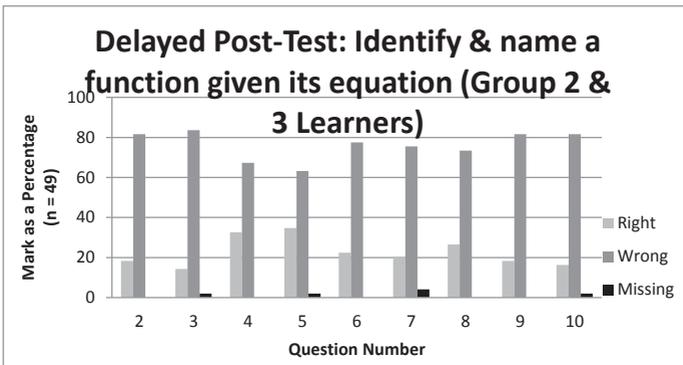


Figure 3: Post-test: identifying a function given its equation – Group 2 and Group 3 learners combined

Figure 2 illustrates group 1 and group 4 learners’ performance on questions that required them to identify and name a function from its algebraic representation. It illustrates that, across the questions, between 40% and 60% of the learners were able to correctly identify the class of function being represented by the given equation. Figure 3 illustrates that, across all the questions, between 60% and 85% of the learners from groups 2 and 3 were **unable** to correctly identify and name the function given its algebraic representation. These two bar charts show that when the critical feature of the object of learning is in focus for the teacher (Figure 2) they are able to provide opportunities for learners to discern the object of learning and this is evidenced by a significant improvement in learners’ performance in relation to questions pertaining to the object of learning.



## SHIFTS IN TEACHERS' THINKING AS A RESULT OF PARTICIPATING IN THE STUDY

Prior to her participation in this study Mampotse saw the textbook as a resource in terms of it guiding her in the sequencing of her lessons and acting as a source for examples. She saw the role of examples as merely providing learners with exposure to different conditions under which a concept has to be recruited when solving a problem. In responding to what informed her choice of examples prior to the study, she referred to the level of the learners, where 'level' refers to how well learners perform in mathematics assessments. For learners who perform well, Mampotse selects more complex examples from the textbook and for weaker learners she selects less complex examples. She also indicated that previously she selected examples that ranged from the simplest to the most difficult and that she chose the examples randomly from the textbook.

After her participation in this study Mampotse indicated that: "this learning study has helped me in discovering that the object of learning should be the driving force for any lesson [...] I have been made aware of the fact that examples do play an important / if not [a] major role in the planning and teaching of the lesson". She now believes that "the examples used should be such that they tend to reinforce the object of learning and are chosen in such a way that they make the learners discern the object of learning". She also sees value in working collaboratively and commented that the "community of practice should be able to periodically meet to iron out errors, misconceptions and problems encountered by teachers in the teaching of different mathematical topics". This comment reinforces that as a result of Mampotse participating in this study, she now attaches value to working with learner error or misconceptions as the point of departure in planning a lesson – we think about and plan a lesson (including the selection of examples) with an object of learning in mind.

Taona also indicated that he used the textbook to guide him with the content that had to be taught and that the textbook was a source from which to choose examples and he supplemented this with past exam papers. He also took into account the level of his learners in terms of their performance in the various assessments when choosing examples for use during the lesson and for homework purposes. He saw the role of examples as a means by which to explain the content that he had to teach, and after the lesson the examples given to learners as homework were for practice and for purposes of establishing for themselves the degree to which they understood the concepts taught.

After participating in the study, he indicated that: "it was an eye opener in as far as the relationship between planning and lesson execution is concerned". He went on to explain this as follows: "I have learnt that in planning one has to be clear of the steps to follow and the sequencing of activities.

Examples used should be in some chronology that builds up to the full attainment of the lesson objectives”. The chronology of the examples is the use of variation to structure and sequence of the examples so that the resulting example space brings the object of learning into focus. He also indicated that what he had learnt was that the “object of learning should always be the reference point in the whole process of teaching”. He now sees the need to ensure that when examples are selected they must be done with the object of learning in mind and that the examples should be carefully sequenced during the delivery of the lesson.

## CONCLUSION

What this learning study managed to achieve was to place on the table and so bring into focus a critical feature that would facilitate an improvement in learners’ ability to identify and name different classes of functions, given their algebraic representation. It also demonstrates a practice in which teachers become deliberate about identifying learner difficulty and deliberately intervene to attend to the learner difficulty.

## REFERENCES

- Marton, F. (2006). Sameness and Difference in Transfer. *The Journal of the Learning Sciences*, 15(4), 499-535.
- Marton, F., & Booth, S. (1997). *Learning and Awareness*: Lawrence Erlbaum Associates.
- Marton, F., & Pang, M. F. (2006). On Some Necessary Conditions of Learning. *The Journal of the Learning Sciences* 15(2), 193 - 220.
- Marton, F., Runesson, U., & Tsui, A. B. M. (2004). The space of learning. In F. Marton & A. B. M. Tsui (Eds.), *Classroom discourse and the space of learning*. Mahwah, New Jersey: Lawrence Erlbaum Associates.
- Stigler, J. W., & Hiebert, J. (1999). *The Teaching Gap: Best Ideas from the World's Teachers for Improving Education in the Classroom* New York: The Free Press.



## EFFECTIVE INTEREST RATES AND BANK ADVERTS: MAKING SENSE OR MAKING CENTS?

Craig Pournara

Wits School of Education, University of Witwatersrand

*Effective interest is an important concept for personal finance. In this paper the concept is defined and illustrated by means of numerical comparisons with nominal rate. A derivation of the formula is provided and the concept is applied to advertised rates of bank products. It appears that these banks advertise average effective annual rates rather than the effective annual rate determined by the conventional formula.*

### INTRODUCTION

The concept of effective interest is important because it enables us to compare different interest rates over different periods, and with different compounding frequencies. Consequently it has important practical implications in the context of personal finance. In the school curriculum, effective interest is introduced in Grade 11 Mathematics and Mathematical Literacy but text books generally do not extend their discussion to engage with the relationship between effective rates and actual banking products. Based on personal experience and discussions with teachers, both teachers and learners find the concept of effective interest elusive and therefore difficult. However, there appears to be no research in this area, which is not surprising given the dearth of research into teaching and learning financial maths in general. In this paper I explore the notion of effective interest, by focusing on its definition, its connection to percentage increase, and the derivation of the effective interest formula. I then discuss the interest rates of notice deposits as advertised by two of the big South African banks and show how effective rates are used in each case.

### Defining effective interest rate

An effective annual interest rate refers to the annual rate of interest of an investment when compounding occurs more often than once a year. It therefore takes into consideration the effect of compounding (Cussen, 2013; Investopedia, 2014). Definitions of effective interest typically refer also to nominal rate. For example, the following definition is provided in the National Curriculum Statement:

Effective interest – the annual rate which is equivalent to a **nominal rate** when compounding is effected more often than once a year (e.g. 12% p.a. compounded monthly is equivalent to 12,68% p.a.; the nominal interest rate ( $i$ ) is 0,12 and the effective interest rate is 0,1268).

(Department of Education (DoE), 2003, p. 87, emphasis in original)

Many definitions provide a formula for calculating effective annual rate. In order to do so, it is first necessary to define some terms.

If  $i$  is the nominal interest rate per annum, then  $i_m = \frac{i}{m}$  represents the effective rate per period, where  $m$  is the number of compounding periods in a year. Thus  $i_4$  is an effective quarterly interest rate,  $i_{12}$  is an effective monthly interest rate, and  $i_1$  is an annual interest rate with annual compounding. In this case the nominal and effective rates are the same. Based on these definitions of terms, we have the following formula for effective annual interest rate:  $i_e = \left(1 + \frac{i}{m}\right)^m - 1$ .

### Making sense of effective interest through numerical examples

It is helpful to illustrate the notion of effective interest by means of examples, and in contrast to nominal interest. Consider the scenario where I invest R800 for a year at 6% p.a. compounded monthly. Each month I receive 0.5% interest on the latest balance. This means that each month I get more interest than the previous month because 0.5% is calculated on a slightly larger amount each time. By contrast, if we consider a simple interest scenario then I still get 0.5% each month but this is always calculated on the original balance of R800.

In Table 1 I compare these 2 scenarios. I include month zero to indicate the starting amount for the 12-month period. In the simple interest section I get R4 interest each month and this accumulates to R48 over the year. In the compound interest section, the interest amount increases each month, and the total interest is R49.34.

Simple interest			Compound interest		
End of month	Balance	Interest	End of month	Balance	Interest
0	800.00		0	800.00	
1	804.00	4.00	1	804.00	4.00
2	808.00	4.00	2	808.02	4.02
3	812.00	4.00	3	812.06	4.04
4	816.00	4.00	4	816.12	4.06
5	820.00	4.00	5	820.20	4.08
6	824.00	4.00	6	824.30	4.10
7	828.00	4.00	7	828.42	4.12
8	832.00	4.00	8	832.57	4.14
9	836.00	4.00	9	836.73	4.16
10	840.00	4.00	10	840.91	4.18
11	844.00	4.00	11	845.12	4.20
12	848.00	4.00	12	849.34	4.23

Table 1: Comparison of simple and compound interest each month for 12 months

I now show the connection between this accumulated amount, percentage increase and the effective annual rate. If we calculate the percentage increase on the principal amount in both cases we get:



$$\text{SI increase} = \frac{848-800}{800} \% = 6\% \quad \text{CI increase} = \frac{849.34-800}{800} \% = 6.1675\%$$

It is important to note that the amount of R849.34 has been rounded to 2 decimal places. If we work to higher levels of accuracy, the percentage increase will be 6.167781% (rounded to 6 decimal places).

Now since the effective rate is an annual interest rate that will make R800 grow to R849.34 in one year, we use the compound interest formula as follows to determine the effective rate:

$$A = P(1 + i)^n$$

$$849.34 = 800(1 + i)^1$$

$$i = \frac{849.34}{800} - 1 = 1.06167781 \dots - 1 = 6.167781\%$$

This is the same answer as that obtained from the percentage increase calculation, which is not surprising since  $i = \frac{849.34}{800} - 1$  is an equivalent form of that calculation. Thus we see that the effective annual rate is the same as percentage increase over a one year period.

### Deriving a formula for effective rate

In many school texts the formula for effective rate is simply stated in a form similar to that given earlier in the paper. Consequently it may appear intimidating to learners. However, it may be less intimidating if learners are aware of its derivation which begins with equating two versions of the compound interest formula.

We begin with a general form of the compound interest formula with  $m$  compounding periods per year:  $A_1 = P \left(1 + \frac{i}{m}\right)^m$ . When we work with effective rate, we compound once at the end of the year. So we have  $A_2 = P \left(1 + i_e\right)^1$ . Now since the two calculations must produce the same outputs, we can equate the two equations:  $P \left(1 + \frac{i_e}{1}\right)^1 = P \left(1 + \frac{i}{m}\right)^m$ . Since  $P$  is common we get:  $1 + i_e = \left(1 + \frac{i}{m}\right)^m$  which leads us to:  $i_e = \left(1 + \frac{i}{m}\right)^m - 1$ . From the formula we can deduce that the effective rate is independent of the principal amount but dependent on the number of compounding periods. Testing our earlier scenario with R800 invested at 6% compounded monthly, we get  $i_e = \left(1 + \frac{0.06}{12}\right)^{12} - 1 = 6.16778118\dots\%$  as expected.

### Application to bank interest rates

In this section I consider two adverts for fixed period investments from two of the big South African banks. When banks advertise notice deposits, they are required to indicate whether the advertised rate is a nominal or an effective rate. They also generally offer different rates depending on the period of the investment, and may offer different rates depending on when interest is paid out.

For example, if interest is paid out at the end of the investment (i.e. on maturity), the rate indicated will be higher than if interest is paid out at regular intervals during the life of the investment.

Figure 1 provides nominal and effective rates for a fixed deposit account from First National Bank (First National Bank, 2014). According to the information provided, if an amount is invested for 12 months, the nominal rate is 5.9%, and the effective rate is 6.06%, based on the convention of monthly compounding.

Fixed Deposit Account		
Period	Interest (nominal)	Interest (effective)
1 month	4.00%	4.00%
2 months	4.80%	4.81%
3 months	5.10%	5.12%
4 months	5.10%	5.13%
5 months	5.10%	5.14%
6 months	5.60%	5.67%
7 months	5.60%	5.68%
8 months	5.60%	5.69%
9 months	5.60%	5.71%
10 months	5.60%	5.72%
11 months	5.60%	5.73%
12 months	5.90%	6.06%
13 months	5.90%	6.08%
18 months	6.20%	6.48%
24 months	6.50%	6.92%
30 months	7.00%	7.70%
48 months	7.25%	8.30%
60 months	7.65%	9.20%

Figure 1: FNB advertised rates for Fixed Deposit Account

We can confirm these figures by using the formula:  $i_e = \left(1 + \frac{5.90}{12}\right)^{12} - 1 = 6.06\%$  (to 2 dp). However, if we try to confirm figures for periods shorter than or longer than one year, we may not get the effective rates advertised. For example, consider the case of 3 years: the bank quotes a nominal annual rate of 7% and an effective annual rate of 7.76%. We can check this by adapting the formula derived above. We know that 7% is the nominal rate that will be compounded 36 times over 3 years, so we have:

$A_1 = P \left(1 + \frac{0.07}{12}\right)^{36}$ . We need to determine an effective rate that will be compounded annually for 3 years to obtain the same final amount, so we have  $A_2 = P(1 + i_e)^3$ . Since  $A_1 = A_2$  and the principal amounts are common, we have:

$(1 + i_e)^3 = \left(1 + \frac{0.07}{12}\right)^{36}$ . Solving for  $i_e$  we get  $i_e = 7.23\%$  (to 2 dp). This is the rate which is compounded annually for 3 years and which produces the same amount as 7% compounded monthly for 3 years. But this is not the effective rate quoted by the bank. There is, however, a second possibility for calculating an effective rate.



Here we think about an *average* rate over the 3 years. In other words we determine the total percentage increase over the 3 years and then divide this by 3. This is the same as working with simple interest. So we have  $A_1$  as defined above giving us the total amount accumulated over 3 years. Then we use the simple interest formula over 3 years with an effective rate as follows:  $A_2 = P(1 + 3 \cdot i_e)$ . Once again, since  $A_1 = A_2$  and the principal amounts are common, we have:  $1 + 3 \cdot i_e = \left(1 + \frac{0.07}{12}\right)^{36}$ . Solving for  $i_e$  we get  $i_e = 7.76\%$  (to 2 dp). This is the rate quoted by the bank. So it appears that the bank is quoting average rates for the 3 year period. This rate is higher than the rate calculated previously option because it works off the principal amount rather than from the closing balance at the end of each year. It is not surprising that a bank would choose to quote a higher rate to draw clients. We can show that the same applies for the figures quoted for 6 months and 24 months.

Now consider Figure 2 where Nedbank offers the following rates on an investment product (Nedbank, 2014), and indicates different rates depending on when interest is paid out.

<b>Nedbank Green Savings Bonds</b>				
<b>18, 24, 36 or 60 month deposits, for clients less than 60 years of age.</b>				
<b>Balance</b>	<b>Period of Investment</b>	<b>Interest Monthly</b>	<b>Interest Half-yearly</b>	<b>Interest on Expiry</b>
R1 000 or more	18 months	6.61%	6.70%	6.92%
	24 months	7.20%	7.30%	7.71%
	36 months	7.73%	7.85%	8.66%
	60 months	8.36%	8.50%	10.33%

Figure 2: Nedbank advertised rates for investment product

If interest is paid out monthly, then we are working with a nominal annual rate since no interest accumulates in the account. If interest is paid on expiry, we are working with an effective annual rate. In other words, the interest rate on expiry should generate the same amount of interest as we would get by receiving interest monthly and withdrawing it from the account.

Once again we need to consider both methods of calculating the effective rate. For illustrative purposes I focus on 24 months and 60 months.

24 months

$$(1 + i_e)^2 = \left(1 + \frac{0.072}{12}\right)^{24}$$

$$i_e = 7.442417\% \text{ to 6 dp}$$

60 months

$$(1 + i_e)^5 = \left(1 + \frac{0.0836}{12}\right)^{60}$$

$$i_e = 8.687884\% \text{ to 6 dp}$$

Neither of these figures agrees with the advertised rates for interest paid on expiry.

Below I provide the calculations for an average effective rate:

$$1 + i_e \cdot 2 = \left(1 + \frac{i}{12}\right)^{2 \times 12}$$

$$2i_e = 0.15438729$$

$$i_e = 7.719365\% \text{ to 6 dp}$$

$$1 + i_e \cdot 5 = \left(1 + \frac{i}{12}\right)^{5 \times 12}$$

$$5i_e = 0.516720888$$

$$i_e = 10.334418\% \text{ to 6 dp}$$

If we truncate the answers (to 2 decimal places), we get the quoted rates for interest paid on expiry. Once again, it appears that the bank is quoting average annual rates.

In the case where interest is paid half-yearly, it appears that the quoted rates have also been calculated using an average rate. I illustrate this for the case of 18 months by making the relevant substitutions:

$$1 + \frac{1}{2}i_e = \left(1 + \frac{i}{12}\right)^{0.5 \times 12}$$

$$\frac{1}{2}i_e = \left(1 + \frac{0.0661}{12}\right)^6 - 1$$

$$\frac{1}{2}i_e = 0.033508 \dots$$

So  $i_e = 6.70\%$ , truncated to 2 decimal places as quoted in the advert. In the first line of the above equation, the right-hand side represents the accumulated amount after six months of monthly compounding. The left-hand side must therefore yield the same amount over a six-month period, using simple interest. This can be achieved by halving the annual interest rate which means we obtain 3.35% interest over six months, and this equates to an annual rate of 6.7%. Using the same calculation with the relevant monthly rate, we can obtain all the quoted rates for interest paid half-yearly.

## CONCLUSION

In this paper I have attended to definitions of effective rate and to numerical illustrations that contrast nominal and effective rates. I have discussed the derivation of the formula, showing how it emerges from the compound interest formula, and have applied the formula to two investment products. Based on calculations I have shown that the banks are advertising *average* effective annual rates rather than the effective rate we obtain from the effective annual rate formula used in schools. This suggests it may be useful to teach learners about average effective rates in order to help them make sense of the banking products they will encounter later in their lives.

## References

- Cussen, M. (2013). Understanding interest rates: Nominal, real and effective Retrieved 16/03/2014, from <http://www.investopedia.com/articles/investing/082113/understanding-interest-rates-nominal-real-and-effective.asp>
- Department of Education (DoE). (2003). *National curriculum statement Grades 10-12 (General Mathematics)*. Pretoria: Department of Education.
- First National Bank. (2014). Investment account rates Retrieved 15/03.2014, from <https://www.fnb.co.za/rates/InvestmentAccounts.html>



- Investopedia. (2014). Effective annual interest rate Retrieved 15/03/2014, from <http://www.investopedia.com/terms/e/effectiveinterest.aspx><http://www.investopedia.com/terms/e/effectiveinterest.aspx>
- Nedbank. (2014). Local rates Retrieved 16/03/2014, from <http://www.nedbank.co.za/website/content/interestrates/local.asp>

## HOW WE TEACH TIME IN GRADE 3

Helene Schoeman, Tania Halls and Nicky Roberts

The Grove Primary School, University of Witwatersrand School of Education

*We have identified the measurement topic of “time” to be a problem in our Foundation Phase teaching. We teach it annually, yet our learners perform poorly in time related assessment tasks (in our school’s formal assessments and in our Annual National Assessments), compared to other mathematics topics. So this year we decided to work collaboratively across the four grade 3 classes to try something new. In this short paper we report on this mini lesson-study from the perspective of one of the teachers. We describe her planning (intended intervention) and then report on what happened (the enacted intervention) from the teacher’s perspective. We then share the teacher’s reflections on her lesson, before giving some observations from the mathematical discussions which resulted from this process of sharing and reflection amongst us colleagues.*

### INTRODUCTION

We have identified the measurement topic of “time” to be a problem in our Foundation Phase teaching. We teach it annually, yet our learners perform poorly in time related assessment tasks (in our school’s formal assessments and in our Annual National Assessments), compared to other mathematics topics. So this year we decided to work collaboratively across the four grade 3 classes to try something new. In this short paper we report on this mini lesson-study from the perspective of one of the teachers. We describe her planning (intended intervention) and then report on what happened in the lesson (the enacted intervention), taught by the first author (Helene Schoeman). We then share some of our teacher reflections on our interpretation of this lesson.

### MOTIVATION

In South Africa the curriculum was reviewed and a version called CAPS (Curriculum and Policy Assessment Statement, 2011) was introduced. As part of this process Annual National Assessments were implemented in order to establish the success of the delivery of the curriculum. In the analysis of the mathematics assessment at our school (The Grove), one of the teachers, Helene, became aware that the learners were inexperienced in concepts and understanding of time. In reflecting on this and other practical aspects of measurement, Helene wondered how the concept of time could be introduced earlier in the year in a meaningful, contextual way that would translate to practical use.

At the Grove our learners use the practice of solving number problems through calculations on a number line quite routinely. Helene wondered what learners would make of a reinterpretation of the circle of the clock face to a linear representation.

She wondered if the linear representation would in any way resolve some of the confusion which presents itself with the analogue clock. She had noticed that children often have knowledge about concepts but find it a challenge to employ this when faced with ‘reading’ the time. She wondered how useful it would be for them if we gave the learners an opportunity to investigate hours and minutes on number lines.

### **THEORETICAL FRAMEWORK**

The theoretical framework for this paper draws on two main concepts: variation theory and embodied learning.

Variation theory is a view on the nature of learning which has been developed in relation to learning and awareness in general (Marton and Booth, 1997). This theory of learning has given rise to an approach to research which charts the variations in lived experiences of learners and teacher. In this case we report deliberately on the lived experience of one of the grade 3 teachers (Helene), attending to the shifts in her approach which were evident from her planning (the intended lesson) to the actual lesson (the enacted lesson) and her reflections on this lesson. In so doing, we draw on Mason’s (2002) notion of ‘teacher noticing’ to analyse what the teacher recorded relating to planning, what evidence she collected of the lesson, and how she reflected on its efficacy.

The second concept of embodied learning has been a key aspect informing the teaching approach in relation to the identified problem of reading time. We felt that children needed to experience time in a new way, and to use their own bodies to feel the movement of time. We wanted to attend particularly to the shift from circular motion (as seen on the clock face) to linear motion (which we work with a lot in using empty number lines for calculation). As this was at grade 3 level, we wanted to connect the work on empty number lines (straight lines) to repetitive circular motion required for the 24 hour clock.

### **How We Teach Time**

In this section we present the intended lesson, the enacted lesson and follow this with the teacher’s reflections on the lesson. We conclude the paper with our collegial observations of what this lesson revealed to us during the discussions which emerged from it.

The lesson was conducted in the first term with a class of 26 grade 3 learners in an affluent suburban school. The focus was on the shift from 12-hour time to 24-hour time on both analogue and digital formats. Learners were physically involved in the lesson as they became the 24 points of a clock while holding a rope and standing in a double circle. They then unwrapped the circle to become a straight number line from 0 to 24. They were able to pace out the twenty four hours of the day, and then illustrate this on number lines. Each learner was allocated an hour of the day to illustrate and this became a mural display in the corridor.

As four classes undertook this lesson simultaneously on the sport field, the continuity of time was brought into focus. Each class represented one day, and the four days could be joined end to end to form four days.

## CONCLUSION

In this paper we describe the planned (intended lesson) undertaken by one of the grade 3 teachers (Helene, who is the first author). This is followed by a rich description of the teachers lived experience of the lesson which we present using a combination of photographs, learner produced work and teacher reflective notes which include her feelings about the lesson, and her recollection of learner talk.

In so doing we hope to provide an example of an “innovative” grade 3 mathematics lesson that was undertaken in this well-resourced primary school context. The value in the paper is two-fold. Firstly it presents an approach to teaching time which may be adapted for use in other classroom contexts. This approach is innovative in its attempt to support learners to embody the motion of time on an analogue clock, and to explicitly connect this to learners’ sense of linear motion (which they experience routinely as part of empty number line work in calculations). As such, this paper may be relevant to other Foundation Phase teachers.

Secondly the paper presents a rich description of the teacher’s planning, her account of her own teaching and her reflections on this process. It is of interest to us to consider what it is that the teacher considered worth marking, talking about with colleagues (in terms of her intentions, enacted lesson and her lived experience of this process), and ultimately formally recording in this paper. So we present this paper as an example of a practicing teacher engaging deeply with a reflective process on a single lesson, and report on what this process has revealed to her and her colleagues. We hope that the paper is useful to teachers in training (be this pre-service or further professional development) and to the teacher educators who support such processes of reflective practice.

## ACKNOWLEDGEMENTS

This paper was made possible as part of the Focus on Primary Maths project. This project is funded by ApexHi and administered by Tshikululu Social Investments.



# LESSON PLAN STUDY ON HOW TO CHOOSE EXAMPLES AND PREPARE EXPLANATIONS FOR HYPERBOLA GRAPH IN GRADE 10

Nkosingiphile Shelembe and Sibongile Mashazi

**Jules High School, Gauteng**

*This paper reports on a study that was conducted over a period of three weeks on 'how to plan a lesson to help learners to understand the meaning of the asymptote from the formula and graph of hyperbola'. The main focus of the study was on the meaning of the asymptote when the learners sketch a hyperbola graph as well as deliberate selection and use of examples by teachers. During the period of three weeks, the study was divided into three phases, namely: planning, Laboratory lesson 1 by the first teacher and Laboratory lesson 2 by the second teacher. Even though it's not possible to conduct such lessons every day, but this study gave us a good practice on how lessons should be planned and conducted.*

## INTRODUCTION

In this paper we share what we did in a lesson plan study that was initiated by Wits Maths Connect Secondary (WMCS) Project. Therefore, we will briefly give a description of the project, the framework that we used, the rationale for the chosen focus which was the hyperbolic function, and then the three phases that constituted the three weeks of lesson planning respectively.

### The project

WMCS is a project that is based at the University of the Witwatersrand, Johannesburg in the Wits School of Education. WMCS offers two Maths courses called Transition Mathematics 1 and 2 (TM1 and TM2), to help Maths educators to develop their knowledge in mathematics for teaching. I and my colleague who is writing this paper with me attended TM1 and now we are attending TM2. What we share in this paper is how we experienced one aspect of the professional development and that is the school based component called lesson plan study. To do the lesson plan study we had a framework that was developed by the project.

### The framework

Firstly, the framework enables us to think about the object of learning, that is to say the capability that you want developed in learners by the end of the lesson. To achieve this, the framework brings a teachers attention to the deliberate selection and use of examples to explain a particular concept. The framework also allows us to think about the activities that learners will be engaged in (see appendix 1).

## Functions

We identified the topic of the hyperbolic function as the one we would work with over the three weeks for two reasons. Firstly, in the official examinations paper one has more questions based on functions. Secondly functions are a tool used to solve various complex problems in Maths, Science and Technology environment. We saw it very important therefore for us to focus on this topic and see how best we can teach it in such a way that learners learn. Achieving this would then help learners improve their ability when approaching mathematical problems in general. We now move on to discuss the three phases, namely: the planning phase, laboratory lesson one and lastly laboratory lesson two.

### PHASE 1: PLANNING

#### Participants

There were six Maths educators and three members from WMCS project that were participants during this phase. Two educators that were part of the team are Maths HOD's with more than ten years of experience teaching Maths at FET level and other educators are post level one educator from two different schools.

In the planning session we were guided by a Mathematics Teaching Framework (MTF) discussed above and shown in the appendix. The MTF lesson plan tool was used as guide for choosing examples, preparing explanations, and learner activity.

#### Rationale for choice of examples

In selecting the examples we looked for examples that would address learner misconceptions and learner difficulties in interpreting and drawing the hyperbola graphs during tests or exams. Learners pre-knowledge and misconceptions about hyperbola graph were put into consideration in the process of selecting and choosing examples. Annexure two shows the examples that were planned to overcome those misconceptions and they were mainly selected from Classroom Mathematics Grade 10 authored by Laridon et al (2011).

### PHASE 2: LABORATORY LESSON 1 AND REFLECTION

This was a first lesson of a study and was presented by teacher 1 using the MTF planned in phase 1. I am a post level one educator who attended TM1course with WMCS and I am the one who was responsible to present the first lesson. The duration of a lesson was one hour.

#### 2.1 Lesson presentation 1

##### Participants

Participants in the lesson were 20 Grade 10 Maths learners, observed by five educators and four WMCS project team members. Educators and Wits members were seated at the back of the classroom during the lesson and six of them were part of the planning stage of the study. Tables were arranged in a way that allows the learners to seat in pairs and the learners were also allowed to work in pairs.



As this was a revision lesson, there was no need to teach learners how to draw a hyperbola graph. The aim was to help the learners to understand the meaning and the effect of the asymptote to the hyperbola graph. The learners were given six equations as shown in figure 1 in annexure 2 and they were asked to use table method to draw the graphs. After drawing graphs, learners were given cards with the same six equations and six graphs to match. This was a very interesting exercise to the learners, because learners had to look at their graphs and compare their graphs with the graphs given in the cards. Even the learners who did not get their graph correct at the beginning of a lesson, they managed to find the card matching correct. They used equations and different features of the graphs; see annexure 3.

After doing card matching exercise, learners were given another exercise to compare the first graph with other five graphs. They were looking at what are common features and what are different features between the graphs. During this exercise, the learners' comparisons were based on how the quadrants shifted. They did not talk about the asymptotes when looking at graphs. They only talked about vertical shifts. The only time when they introduced the word 'asymptote' is when they were trying to use equations to explain their observations about graphs. The learners could see the asymptote from the equation, but they only see the lines shifting from the graphs which they spoke about as quadrants shifting.

The last exercise for the day was to sketch the hyperbola graphs without using table method. Even though initially this exercise was not planned to be the last, but due to the amount of time that was remaining, the educator was forced to make it to be the last. The activity in figure 2 was initially planned to be the last. The activity in figure 2 was planned to help the learners to observe what happens to the values of  $y$  as the values of  $x$  get very closed to an asymptote, get very small and get very big. This activity was also useful to help the learners to understand the meaning of the asymptote.

### Graph sketching activity

The sketching exercise went very fast and proved to be the most interesting exercise to learners. Learners' participation and solutions displayed to have an understanding of the asymptote from the equation to the graph. However, when it came to the:  $y = \frac{x}{2} + 1$ . All learners produced a hyperbola graph for this equation. They did not notice that this is no longer a hyperbola graph but it's a straight line graph. It was particularly a useful part of the entire experience because it showed the importance of selecting examples. This marked the end of the lesson, learners left and we remained to reflect and discuss the lesson.

## **2.2 Reflection**

In the reflection and discussion session we focused on what was the object of learning; what examples and explanations were offered. For examples additions were made for the next lesson and those changes from first lesson will appear under lesson 2.

The lesson that we all took from having carefully thought through the selection of examples and preparation of explanations in the planning is that good examples do not only help to expose learners misunderstanding, but also assist in helping learners to understand the meaning around the concept. The reflection and discussion made it possible to reveal learners' misconceptions/ misunderstanding and helped inform our planning for the next lesson that was going to be taught by a different teacher to a different group of grade 10 learners.

### Phase 3: Laboratory lesson 2 and Reflection

Laboratory lesson 2 was a final stage of the study of the lesson preparation and execution. It was therefore the second lesson executed by teacher 2 after we re-planned the same lesson taught by the first teacher. I am a level one teacher who participated in TM1 course offered by WMCS in 2012. The lesson was taught for duration of 1 hour. In this phase, I would elaborate on this lesson referred to as laboratory lesson 2 and report on its reflection.

#### 3.1 Lesson presentation 2

The participants were a new group of 22 Grade 10 learners (who were not involved in lesson 1) from my school, 5 WMCS members, 2 mathematics teachers from a neighbouring school as well as 2 mathematics colleagues. The members from WMCS and teachers observed the lesson by sitting at the back of the classroom (there were instances when they moved around as learners were engaged with class activities). Learners were working in pairs. Prior to the lesson, learners were given homework (same as in lesson 1) to draw the following functions: 1.  $y = \frac{2}{x}$ ; 2.  $y = \frac{-2}{x}$ ; 3.  $y = \frac{2}{x} + 3$  and 4.  $y = \frac{2}{x} - 3$ . The lesson itself was segmented into the following points: (a) general form  $y = q + \frac{a}{x}$ , (b) asymptote and (c) the effect of a and q which I discuss below in that order.

#### (a) Discussion around $y = \frac{a}{x} + q$ and $y = q + \frac{a}{x}$

I began the lesson by asking learners to compare each other's homework. I then gave them the card matching activity as in Figure 1, except that we used  $y = 3 - \frac{2}{x}$  for equation six instead of  $y = -\frac{2}{x} - 3$ . The six graphs for the matching activity were the same as in lesson 1. This activity was a follow up on learners' homework where they were expected to match an equation with a corresponding graph (see Table 1 in the appendix). There were actually five equations because the sixth equation was a repetition of  $y = -\frac{2}{x} + 3$  but written in a form of  $y = 3 - \frac{2}{x}$ . These two equations were included purposefully to explore if learners would be able to observe that they are the same. Hence, one of the graphs had no corresponding equation. When I asked learners what is the equation of the last graph (see Figure 3 below), they all shouted that it was obvious referring to  $y = 3 - \frac{2}{x}$ .



However, one learner noted that the asymptote of the graph is  $-3$  and not  $3$ , while others claimed that it was a computer error. They eventually discovered that  $y = 3 - \frac{2}{x}$  was not a corresponding equation for graph in figure 3, and hence noticed that the two equations were the same. This was an interesting part of the activity because learners are so used to the general form of  $y = \frac{a}{x} + q$  instead of  $y = q + \frac{a}{x}$ . After this activity I then moved on to

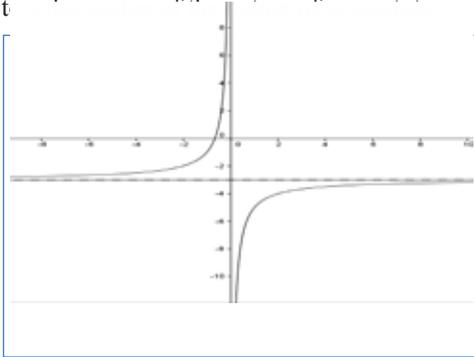


Figure 3: The hyperbola graph of  $y = -\frac{2}{x} - 3$

(b) Discussion on asymptote

I then asked learners to focus on the graphs of equations 1 to 4 (these were functions given to learners for their homework) where I asked them to compare these graphs with respect to the things that changed and the things that are the same. According to Rowland (2008), variation allows learners to differentiate things. I first asked them to compare graph 1 and 3 as agreed in the re-planning session. Learners noted the movement of the graph by three units up, change of the range as well as the change of the horizontal asymptotes. I then engaged learners on the idea of the asymptote where I asked learners how many asymptotes a graph has. In most cases, learners do not recognise the vertical asymptote. They then realised that the vertical asymptote and the position of the graph stayed the same. I highlighted the issue of the position of the graph that we look at it with respect to the asymptotes, instead of shifting quadrants.

One learner noted that the line of symmetry has changed in graph 3 and it is  $y = \frac{2}{x} + 3$ . I did not expect this contribution especially because the learner gave the equation of symmetry. I recognised the learner by asking the class to applaud for her. I continued to ask learners to compare graphs 1 and 4 as well as graph 1 and 2 in terms of what changes and what stays the same.

(c) The effect of “a” and “q”

In the next stage of the lesson we discussed the effect of  $a$  and  $q$  on the graph of  $y = \frac{a}{x} + q$ .

However, learners were the ones to say that 'a' determines the position where the graph appears. They further said that  $q$  is the asymptote and it determines the movement of the graph. I then gave learners three more equations to draw rough sketch on the white board given to them ( $7. y = \frac{4}{x} + 5$ ,  $8. y = 2 - \frac{4}{x}$  and  $9. y = \frac{x}{2} + 3$ ). Learners successfully sketched with the exception of graph number 9. Here only one learner recognised that it was a linear, while the rest drew a hyperbola. This was the last activity for the lesson. Learners were dismissed and we sat for a reflection and discussion session.

### 3.2 Reflection

WMCS members and mathematics teachers reflected on the lesson as soon as learners left. It was noted that the learners learnt what we expected them to learn. This was clearly demonstrated from the last activity. Again, the sequence of comparing graphs worked very well as compared to lesson 1.

We also reflected on the fact that it was a good idea to use six equations and graphs for card matching activity instead of five equations and graphs that we agreed upon during the re-planning session. This gave learners a platform to come up with the equation of the graph..

#### Phase3: Laboratory lesson 2 and Reflection

Laboratory lesson 2 was a final stage of the study of the lesson preparation and execution. Therefore, this was the second lesson executed by myself after we re-planned the same lesson taught by the first teacher. In this phase, I would elaborate on this lesson referred to as laboratory lesson 2 and report on its reflection.

#### Lesson presentation 2

The participants were a new group of 22 Grade 10 learners from my school, 5 Wits Maths Connect (WMC) officials, 2 mathematics teachers from a neighbouring school as well as 2 mathematics colleagues. The officials from WMC and teachers observed the lesson by sitting at the back of the classroom (there were instances when they moved around as learners were engaged with class activity). Learners were working in pairs. Prior to the lesson, learners were given a homework to draw the following functions:  $1. y = \frac{2}{x}$ ,  $2. y = \frac{-2}{x}$ ,  $3. y = \frac{2}{x} + 3$  and  $4. y = \frac{2}{x} - 3$ .

I began the lesson by asking learners to compare each other's homework. I then gave them the card matching activity, which consisted of six equations and six graphs. This activity was a follow up on learners' homework where they were expected to match an equation with a corresponding graph (see Table 1 in the appendix). There were actually five equations because the sixth equation was a repetition of  $y = -\frac{2}{x} + 3$  but written in a form of  $y = 3 - \frac{2}{x}$ . These two equations were included purposefully to explore if learners would be able to observe that they are the same. Hence, one of the graphs had no corresponding equation.



When I asked learners what is the equation of the last graph, they all shouted that it was obvious referring to  $y = 3 - \frac{2}{x}$ . However, one learner noted that the asymptote of the graph is -3 and not 3, while others claimed that it was a computer error. They eventually discovered that  $y = 3 - \frac{2}{x}$  was not a corresponding graph, hence noticed that the two equations were the same. This was an interesting part of the activity because learners are so used to the general form of  $y = \frac{a}{x} + q$  instead of  $y = q + \frac{a}{x}$ .

I then asked learners to focus on the graphs of equations 1 to 4 (these were functions given to learners for their homework) where I asked them to compare these graphs with respect to the things that changed and the things that are the same. I first asked them to compare graph 1 and 3 as agreed in the re-planning session. Learners noted the movement of the graph by three units up, change of the range as well as the change of the horizontal asymptotes. I then build up on the idea of the asymptote where I asked learners how many asymptotes a graph has. In most cases, learners do not recognise the vertical asymptote. They then realised that the vertical asymptote and the position of the graph stayed the same. I highlighted the issue of the position of the graph that we look at it with respect to the asymptotes. One learner noted that the line of symmetry has changed in graph 3 and it is  $y=x+3$ . I did not expect this contribution especially because the learner gave the equation of symmetry. I recognised the learner by asking the class to applaud for her. I continued to ask learners to compare graphs 1 and 4 as well as graph 1 and 2 in terms of what changes and what stays the same.

While comparing graph 1 and 2, the position of the graph seemed to be the main thing that changed. As agreed in the re-planning session that I must refrain from using quadrant, I labelled the Cartesian plane as region 1 to region 4 instead of quadrant 1 to quadrant 4. Hence, I explained to learners why the graph was appearing in region 1 and region 3 for graph 1 as well as region 2 and 4 for graph 2. This was done by re-writing the equations as  $xy = 2$  and  $xy = -2$ . I further elaborated that the product of  $xy$  can only be positive 2 in region 1 and 3. Again, the product of  $xy$  is only negative 2 in region 2 and 4. Majority of learners nodded their head to confirm that it make sense. One of the observers noted that one learner said, “Final someone is making sense and that explanation was needed”. However, it was noted by another observer that this learner wanted to extend the same idea of determining the position of equations 3 and 4 but got stacked. I then concluded this stage of the lesson by asking learners the name of the graph we dealt with as well as its general form.

In the next stage of the lesson I built up on the idea of the general form of the hyperbola in terms of the effect of “a” and “q”. However, learners were the ones to say that ‘a’ determines the position where the graph appears. They further said that q is the asymptote and it determines the movement of the graph.

I then gave learners three more equations to draw rough sketch on the white board given to them ( $7. y = \frac{4}{x} + 5$ ,  $8. y = 2 - \frac{4}{x}$  and  $9. y = \frac{x}{2} + 3$ ). I asked them to put up their boards so that everyone would see their sketch. Furthermore, they were asked to draw each sketch on the chalkboard. Although majority of learners got the sketch of equation 7 and 8 correctly, but equation 9 seemed to be a challenge because they drew a hyperbola. However, one learner noticed that the equation was linear as she said, “ $x$  appears in the numerator not denominator”. This indicated that learners were used to the general form but did not put the attention on the position of  $x$ . I then concluded the lesson by asking learners to define an asymptote.

### **Reflection**

WMC and mathematics teachers conducted reflection immediately after the lesson. It was noted that the learners learnt what we expected them to learn. This was clearly demonstrated when they were sketching graphs on the chalkboard. Therefore, the lesson went well especially because learners contributed more than what we expected. Again, the sequence of comparing graphs worked very well as compared to lesson 1. It was noted that the position of the graph came up clearly, because teacher 2 explained it with respect to the asymptotes. However, it was easy for learners to grasp the explanation of graph position when ‘ $q$ ’ was not included but this explanation was not extended for equations with ‘ $q$ ’. It was thus concluded that in future the lesson should incorporate example with the value of ‘ $q$ ’ when explaining why the position of the graph appears in certain quadrants. It was also noted that introducing regions instead of quadrants would create a misconception especially when learners progress to the next grade and are taught by someone else.

We also reflected on the fact that it was a good idea to use six equations and graphs for card matching activity instead of five equations and graphs that we agreed upon during the re-planning session. This gave learners a platform to come up with the equation of the graph. Again, it allowed them to work with equations in a form of  $y = \frac{a}{x} + q$  and  $y = q + \frac{a}{x}$ . However, it was noted that some learners said ‘3’ is a slope and  $\frac{-3}{x}$  is an asymptote. This raised a question of what is learners’ explanation of swapping  $\frac{a}{x}$  and ‘ $q$ ’.

### **CONCLUSION**

This was a very good exercise for our development as teachers. Inputs from different people were very useful in terms of influencing and shaping our thinking around the deliberate selection and use of examples and careful preparation of explanations. The planning phase gave us confidence to present the lessons and the reflection sessions indicated that more changes needed to be done in the same lesson. This indicated to us that there will never be a time where you come to a point of a good lesson that needs no improvement; there will always be space for comment and more improvement. Therefore, we learnt that the process of lesson planning is continuous.



This means that every time one plans a lesson you must expect changes because we are teaching learners with different thinking abilities. It would be great if such studies are implemented in other schools. Henceforth, we look forward to see these kinds of lessons taking place more often during 2014. This was a good exercise for teachers to develop and also for the learners to learn better.

### **ACKNOWLEDGEMENTS**

We would like to thank the WMCS project team for instilling the positive spirit of planning a lesson and also for taking their time to come to our school. We would also like to thank Nontsikelelo Luxomo for helping us write the paper.

### **REFERENCES**

*Laridon, P. et al, (2011).* Classroom Mathematics Grade 10 Learner's Book, Heinemann, Sandton

Rowland, T. (2008). The purpose, design and use of examples in the teaching of elementary mathematics. *Educational Studies in Mathematics*, 69, 149-163.

**Annexure 1**

Use table method to draw the graphs of the following functions:

1.  $y = \frac{2}{x}$

4.  $y = \frac{2}{x} - 3$

2.  $y = \frac{-2}{x}$

5.  $y = \frac{-2}{x} + 3$

3.  $y = \frac{2}{x} + 3$

6.  $y = \frac{-2}{x} - 3$

Use the equation:  $y = \frac{2}{x}$  to complete the following tables:

(a)

x	0.01	0.001	0.0001	0.00001	0.000001
y					

(b)

x	- 0.01	- 0.001	- 0.0001	- 0.00001	- 0.000001
y					

(c)

x	100	1000	10 000	100 000
y				

(d)

X	- 100	- 1000	- 10 000	- 100 000
y				



## CONNECTING MATHEMATICS WITH ITS HISTORY

Avhasei Richard Tsanwani

Mastec Institute (Continuing Professional Development: LDoE)

*The purpose of this paper is to illustrate how historically based activities can be used in the teaching and learning of mathematics. Some illustrative examples of connecting mathematics with its history are presented. It is not a comprehensive paper, but is rather intended to discuss ideas that the teachers can use or modify depending on the particular needs and abilities of their learners at various mathematics levels. It is hoped that teachers can get more instructional insight from reading this paper.*

### INTRODUCTION

The history of mathematics, including its principles, procedures, and personalities, is often one of the most neglected areas in our teaching of mathematics. However, one of the specific aims for Curriculum and Assessment Policy Statement (CAPS) for mathematics in South Africa is to show mathematics as a human activity by including the history of mathematics (Department of Basic Education, 2011). Internationally, there is an overwhelming agreement among mathematics educators on the importance of introducing history of mathematics into the school mathematics curriculum. In this regard, the National Council of Teachers of Mathematics (NCTM) initiated a professional development scholarship emphasizing the History of Mathematics (NCTM, 2011). The goal of NCTM is to: *provide financial support for:*

- *completing credited course work in the history of mathematics,*
- *creating and field-testing appropriate classroom activities incorporating the history of mathematics, and*
- *preparing and delivering a professional development presentation to colleagues. (NCTM, 2011)*

Quite simply, these statements affirm the importance of connecting mathematics with its history.

According to Swetz (1994), many learners' think of mathematics as a very dull subject. The only reference to history in the mathematics class is by those teachers whose enthusiasm for mathematics has extended their knowledge to its historical development. Even then, reference to the history of mathematics is done at leisure depending on the teacher's pedagogic style.

There are various strategies that can be used for introducing a historical perspective into the teaching of mathematics:

- Using historically based films or videotapes in classroom instruction.
- Assigning classical or historical problems and noting their origins or significance.
- Obtaining information about the origins and meanings of mathematical terms, symbols, and words.
- A consideration of the people of mathematics – their lives and work of selected mathematical personages.

In this regard, two examples historically based activities will be shown in this paper. It is not a comprehensive paper, but is rather intended to discuss ideas that the teachers can use or modify depending on their particular needs and abilities of their students at various mathematics levels.

There are various ways in which the examples cited in this paper can be conducted. This will depend on individual teacher's pedagogical preference. For instance, a single activity can be undertaken by the class as a whole; or learners can be divided into small groups for a specific activity or single one learner can undertake individual activities, either as an investigation or project. Each example provided in this paper begins with a historical introduction to the topic. Although much of the work can be completed during class time, all activities in this paper include suggested exercises at the end to test learners' understanding. Teachers' journals and periodicals occasionally carry articles on the history of mathematics and its use in teaching. And so for more activities, teachers are advised to consult them, and more importantly, books on the history of mathematics.

## **EXAMPLE1: FINDING THE PRODUCT OF NUMBERS**

### **INTRODUCTION**

In Medieval Europe the possession of a computing device such as an abacus and knowledge of the rules necessary to use it were limited to a few people, called masters of abacus or reckoning masters. Many of these men were merchants, and it is from this group that the concept of a professional mathematician arose. The master's jealousy guarded their special knowledge, sharing only with a few selected apprentices. These apprentices paid handsomely to learn the skills and arts of a mathematician. Gradually, from the 13<sup>th</sup> century onward, a new set of symbols; the Hindu-Arabic numerals were used to represent numbers. Now almost everyone could learn how to perform numerical calculations. We are going to examine one of the new ways people in the 15<sup>th</sup> century learned to do multiplication.



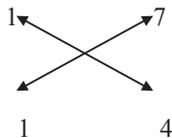
### ACTIVITY 1: (MULTIPLICATION ALGORITHM OF TWO-DIGIT NUMBERS)

Consider the table given below:

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

This table of numbers is known as Pythagorean table. Historically it was used among others to find the product two numbers. For example, the products of the numbers in the first row and first column can be read at the intersection of the row and column e.g.  $7 \times 6 = 42$ . Furthermore, the diagonal line on this table represent the fact that half of the table above the line is the same as half the of the table below the line (Symmetrical).

Now, let's consider another multiplication algorithm of two-digit numbers from Italia called per croceta, meaning "by the cross". Let's find the product of 17 and 14.



**PROCEDURE:**

First multiply 4 by 7. (A: 28). Write down the 8 and retain the 2. “by the cross”, multiply 1 by 4 and 7 by 1. (A: 4 and 7), Add the results. (A:  $4 + 7 = 11$ ). Add the 2 retained from the previous computation. (A:  $11 + 2 = 13$ ). Write the 3 to the left of 8. (A: 38). Finally, multiply the 1’s together. (A:  $1 + 1 = 2$ ). Add the 1 retained from the previous computation. (A:  $1 + 1 = 2$ ). Write this 2 to the left of 3. (A: 238)

**ACTIVITY 2: (Finding the product of numbers with more digits.)**

15<sup>th</sup> century mathematics employed several different schemes or algorithms in this regard. The word ‘algorithms’ comes from the Latin rendering of the name of the great Arabic mathematician al-khwarizmi (Ca. 800-8470), the author of the principal works that exposed Europeans to the Hindu-Arabic numerals and their methods of computation. One of the most popular 15<sup>th</sup> century algorithms for multiplication was the gelosia method.

**PROCEDURE:**

(a) Finding the product of  $372 \times 431$ :

- One number (372) must be written across the top of the grid. The other number (431) is written down the right side of the grid.
- Compute the product of each row number and column number and write in the particular cell shared by the row and the column. Write the individual products with their tens digit above the diagonal line and their units digit below the line.

	3	7	2	
1	2	0	4	
2	8	8		
0	2	0	3	
9	1	6		
0	0	0	1	
3	7	2		

- Starting at the lower right corner of the grids, sum the entries along each diagonal path within the grid. Write the units result at the end of each diagonal path and carry the tens digit to the path above and proceed in the same manner. When you are finished summing, read the product along the left and bottom edge of the grid: 160,332



	3	7	2	
↑	1 2	2 8	0 8	4
	0 9	2 1	0 6	3
	0 3	0 7	0 2	1
	→			

### EXAMPLE 2: NUMBER SHAPES

Greece produced great mathematicians whose work cannot be fully covered here. One of such great mathematicians is Pythagoras, who was born about 580 B.C. on the Island of Samos in the Aegean Sea. His teachings attracted student disciples who formed into a brotherhood of fraternity. These disciples were called Pythagoreans. They were obsessed with the concept of numbers; for the “all was number”. They used geometric shapes to represent various whole numbers. Today such numbers are called polygonal or figurate numbers. These are numbers which are associated with geometric figures.

#### PROCEDURE:

Let's consider three classes of figurate numbers:

- Triangular numbers: These are numbers represented by a triangle of dots. We will denote it by  $T_n$ , where  $n$  indicates the order of the number, i.e.  $T_4 = 10$

Example:  $T_4 = 10$



This is number 10. (Count them. Rather, what exists is a system of numeration, presently the decimal system of positional numeration.

- Square numbers: these are numbers represented by a square of dots, denoted by  $S_n$ ,  
Example:  $S_2 = 4$

o	o
o	o

- Oblong numbers: These are numbers represented by rectangular dots, that has one more column than in rows. They are denoted by  $O_n$ ,  
Example:  $O_4 = 20$

o	o	o	o	o
o	o	o	o	o
o	o	o	o	o
o	o	o	o	o

1. Complete the table given below to find the next few triangular, square, and oblong numbers:

	Square Number	Triangular Number	Oblong Number
1st			
2nd			
3rd			
4th			
30th			

2. Use the table above to find the following:
  - (a)  $T_n =$  nth triangular number (Hint: draw a rectangle made up of two triangular numbers of the same size).
  - (b)  $S_n = T_n + T_{n-1}$

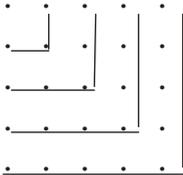
Verify that the sum of the two consecutive triangular numbers is a square number, and find  $S_n =$  nth square number).



(c)  $O_n =$  nth oblong number.

(d) The sum of the nth oblong number and the nth square number.

3. Consider a pattern found in a square number:



The number represented by each L configuration is an odd number. Use this pattern to find the sum on  $n$  consecutive odd positive integers.

4. The Greek mathematician Archimedes (287 – 212 B.C.) developed a formula for the sum of the square numbers:

$$S_1 + S_2 + S_3 + \dots + S_n = \frac{n(n+1)(2n+1)}{6}. \text{ Verify this formula.}$$

#### EXERCISES:

(a) Complete the table below to help you investigate other number shapes. You should find that there are many relationships among the various number shapes. Look for vertical and horizontal patterns that can help you.

	1st	2nd	3rd	4th	5th	6th	7th
<b>Triangular</b>	1	3	6	10			
<b>Square</b>	1	4	9				
<b>Pentagonal</b>	1	5	12				
<b>Hexagonal</b>	1	6	15				
<b>Heptagonal</b>	1						

## CONCLUSION

Teachers agree in principle, but many of them are concerned about adding yet another unit to an already crowded curriculum. It is my belief that including a historical dimension in the mathematics classroom does not mean replacing any part of the mathematics curriculum. We can include the historical perspective in our mathematics classroom lessons naturally. We do not have to view this as something extra, but rather as a complement to mathematics learning, a background, a perspective that helps clarify the topic under discussion. Students must clearly see the connections between the mathematics they are studying and are relevant historical facts.

## REFERENCES

1. Department of Basic Education (2011). *Curriculum and Assessment Policy Statement (CAPS) Grades 10–12: Mathematics*. Pretoria: Department of Basic Education.
2. Department of Education. (2003). *National Curriculum Statement: Grades 10–12 (General): Mathematics*. Pretoria: Department of Education.
3. Peggy A.H and Arthur F.C. (1995). *NCTM Yearbook: Connecting Mathematics Across the Curriculum*. Reston, VA: National Council of Teachers of Mathematics.
4. National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: Author.
5. Swetz, F. J. (1994). *Learning activities from the history of mathematics*. Portland, ME: J. Weston Walch.

